*By the same authors*

## TRIGONOMETRY

PART I. INTERMEDIATE TRIGONOMETRY, 6*s*.

PART II. HIGHER TRIGONOMETRY, 5*s*.

PART IV. SPHERICAL TRIGONOMETRY, 3*s*. 6*d*.

# TRIGONOMETRY

## PART III
## ADVANCED TRIGONOMETRY


*by*

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# PREFACE TO PART III

Part III of this book deals mainly with the theory of the convergence of series and of products, and with the applications of that theory to the trigonometrical functions.

In Chapter XVIII an elementary account of convergence of series is given. This is followed by a chapter on uniform convergence. Chapter XX is chiefly concerned with infinite products and with functions of a complex variable, these functions being defined by means of series of complex terms. This chapter also includes some applications of the theory of Dirichlet's Integrals to trigonometrical series. Large collections of examples, with answers, will be found at the ends of the chapters. A discussion of the length of a circular arc is contained in an Appendix.

We have again to thank Mr. Albert Anderson for his valuable help with proof correction.

T. M. M.
W. A.

# CONTENTS

## CHAPTER XVIII

## ERRATA

p. 395 footnote :

*For* $u_n(x)$ *read* $u'_n(x)$

p. 435 :

*Replace lines 2–4 by*

$$\frac{\sin 2\theta}{2\sin\theta}=\frac{\mathop{\mathcal{L}}\limits_{n\to\infty} 2\theta \prod\limits_{r=1}^{2n}\left(1-\dfrac{4\theta^2}{r^2\pi^2}\right)}{\mathop{\mathcal{L}}\limits_{n\to\infty} 2\theta \prod\limits_{r=1}^{n}\left(1-\dfrac{4\theta^2}{4r^2\pi^2}\right)}$$

$$=\mathop{\mathcal{L}}\limits_{n\to\infty}\prod_{r=1}^{n}\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\},$$

and therefore

$$\cos\theta=\prod_{r=1}^{\infty}\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}.$$

## Chapter XX

## INFINITE PRODUCTS: FUNCTIONS OF A COMPLEX VARIABLE



## Appendix

# TRIGONOMETRY

## PART III

## ADVANCED TRIGONOMETRY

# CHAPTER XVIII

# CONVERGENCE OF SERIES

## § 1. Convergence of Sequences

In Chapter XV, § 9, the condition that a sequence $(u_n)$ should converge to a limit $l$ was given.

*Example* 1.—If the sequence $(u_n)$ converges to $l$, show that the sequence $(cu_n)$, where $c$ is a constant, converges to $cl$.

If $c \neq 0$, choose $m$ so large that, if $n \geqq m$, $| u_n - l | < \epsilon/| c |$. Then, if $n \geqq m$, $| cu_n - cl | < \epsilon$. The result for $c = 0$ is obvious.

*Example* 2.—If the sequences $(u_n)$ and $(v_n)$ converge to $l$ and $k$ respectively, show that the sequences $(u_n + v_n)$ and $(u_n v_n)$ converge to $(l + k)$ and $lk$ respectively.

Choose $m_1$ and $m_2$ so large that, if $n \geqq m_1$, $| u_n - l | < \frac{1}{2}\epsilon$ and, if $n \geqq m_2$, $| v_n - k | < \frac{1}{2}\epsilon$. Then, if $m$ is the larger of $m_1$ and $m_2$, and if $n \geqq m$,

$$| (u_n + v_n) - (l + k) | \leqq | u_n - l | + | v_n - k | < \epsilon.$$

Again,

$$u_n v_n - lk = (u_n - l)(v_n - k) + k(u_n - l) + l(v_n - k),$$

and each term on the R.H.S. tends to zero when $n$ tends to infinity. Hence the L.H.S. tends to zero, so that $u_n v_n$ tends to $lk$.

*Example* 3.—If, in Example 2, $k$ is not zero, show that the sequences $(1/v_n)$ and $(u_n/v_n)$ converge to $1/k$ and $l/k$ respectively.

The sequence $( | v_n | )$ converges to $| k |$. Choose $m$ so large that, if $n \geqq m$,

$$\big| | v_n | - | k | \big| < \tfrac{1}{2} | k |.$$

Then

$$| v_n | > | k | - \tfrac{1}{2} | k | = \tfrac{1}{2} | k |,$$

so that

$$\left| \frac{1}{v_n} - \frac{1}{k} \right| = \left| \frac{v_n - k}{v_n k} \right| < \frac{2}{| k |^2} | v_n - k |.$$

But, when $n$ tends to infinity, $v_n - k$ tends to zero: thus $(1/v_n - 1/k)$ tends to zero, or $1/v_n$ tends to $1/k$.

The second result then follows from the second part of Example 2.

It may happen that, while a sequence is convergent, the value of the limit is not known. For such cases new tests of convergence, not involving a knowledge of the limit, are required. There are three fundamental tests of this kind, given in the three theorems below. The following definitions are required:

*Bounded Sequences.*—A sequence $(u_n)$ is said to be *bounded above* if, for all values of $n$, $u_n \leqq k$, where $k$ is finite and independent of $n$. If $u_n \geqq k$ the sequence is *bounded below.* A sequence which is bounded both above and below is called a *bounded sequence.*

*Note.*—A convergent sequence is bounded.

Let $(u_n)$ be the sequence, and let $l$ be its limit. Then, corresponding to any assigned positive number $\epsilon$, an integer $m$ can be found, such that if $n \geqq m$,

$$l - \epsilon < u_n < l + \epsilon.$$

Again, let A be the greatest and B the least of the numbers $u_1, u_2, \ldots, u_{m-1}$. Then, if M is the greater of A and $l + \epsilon$, and $m$ the lesser of B and $l - \epsilon$,

$$m \leqq u_n \leqq M$$

for all values of $n$.

*Monotonic Sequences.*—If, for all values of $n$, $u_{n+1} \geqq u_n$, the sequence $(u_n)$ is called a *monotonic increasing sequence.* If, for all values of $n$, $u_{n+1} \leqq u_n$, the sequence is *monotonic decreasing.*

THEOREM I.—If the sequence $(u_n)$ is monotonic increasing and bounded above, $u_n$ being $\leqq k$ for all values of $n$, it converges to a limit $l$, where $l \leqq k$.

If a monotonic increasing sequence is not bounded above, it diverges to $+\infty$.

For a full discussion, based on arithmetical conceptions, of this and the two following theorems, the reader is referred to Gibson's "Advanced Calculus," Chapter II, or to Bromwich's "Infinite Series," Appendix I. The theorems may, however, be verified geometrically, subject to the assumption that there is a complete correspondence between the system of real numbers, rational and irrational, and the points on an $x$-axis, the numbers being the abscissæ of the corresponding points.

Let $P_n$ (Fig. 1), where $n$ is any positive integer, be the point on the $x$-axis whose abscissa is $u_n$. Then all the points on the axis belong to one of two classes. The upper class consists of those points which lie to the right of every $P_n$, and includes all points to the right of the point K whose abscissa is $k$. The lower class consists of every point $P_n$ and all points which lie to the left of any $P_n$. Every point


FIG. 1.

on the axis belongs to one of these classes, and every point in the lower class lies to the left of every point in the upper class. Let L be the point which separates the one class from the other; sometimes L belongs to the lower class, sometimes to the upper. Then $l$, the abscissa of L, is the limit of the sequence. For, no matter how small a positive number $\epsilon$ may be, there must be an element $u_m$ of the sequence greater than $l - \epsilon$, or the point whose abscissa is $l - \frac{1}{2}\epsilon$ would belong to the upper class. Hence $l - u_m < \epsilon$, and therefore, for every $n \geqq m$, $| l - u_n | < \epsilon$, so that $l$ is the limit of the sequence.

*Example* 4.—If $u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$, show that the sequence $(u_n)$ is convergent.

$$u_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} < 3.$$

Thus the sequence is bounded above. But it is monotonic increasing. Hence it is convergent.

*Example* 5.—If $u_n = \left(1 + \frac{1}{n}\right)^n$, show that the sequence $(u_n)$ is convergent.

$$u_n = 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \ldots$$
$$\ldots + \frac{n(n-1)\ldots\{n-(n-1)\}}{n!}\frac{1}{n^n}$$
$$= 1 + 1 + \left(1 - \frac{1}{n}\right)\frac{1}{2!} + \ldots$$
$$\ldots + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)\frac{1}{n!}.$$

Similarly,

$$u_{n+1} = 1 + 1 + \left(1 - \frac{1}{n+1}\right)\frac{1}{2!} + \ldots$$
$$\ldots + \left(1 - \frac{1}{n+1}\right)\ldots\left(1 - \frac{n-1}{n+1}\right)\frac{1}{n!}$$
$$+ \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\left(1 - \frac{n}{n+1}\right)\frac{1}{(n+1)!}.$$

Now each term in the second series is equal to or greater than the corresponding term in the first series, and the additional term in the second series is positive. Thus $u_{n+1} > u_n$.

Also

$$u_n < 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!} < 3.$$

Hence the sequence is monotonic increasing and bounded above. It is therefore convergent. From formula (29), Chapter XVII, we see that the limit is $e$.

Theorem II.—If the sequence $(u_n)$ is monotonic decreasing and bounded below, $u_n$ being $\geqq k$ for all values of $n$, it converges to a limit $l$, where $l \geqq k$.

If a monotonic decreasing sequence is not bounded below, it diverges to $-\infty$.

Theorem II can be proved in much the same manner as Theorem I; or it may be deduced from Theorem I as follows:

Let $v_n = -u_n$, for all values of $n$. Then the sequence $(v_n)$ satisfies the conditions of Theorem I, and consequently it converges to a limit, $h$ say. Hence, if $l = -h$, the sequence $(u_n)$ converges to $l$.

The condition for convergence enunciated in Theorem III is known as *The General Principle of Convergence.*

THEOREM III.—The necessary and sufficient condition that a sequence $(u_n)$ should converge to a definite limit is that, corresponding to any arbitrarily assigned positive quantity $\epsilon$, however small, a positive integer $m$ can be found such that, if $n \geqq m$,

$$| u_{n+p} - u_n | < \epsilon,$$

where $p$ is any positive integer.

The condition is necessary. For, if the sequence converges to a limit $l$, an integer $m$ can be found such that, if $n \geqq m$,

$$| u_n - l | < \tfrac{1}{2}\epsilon.$$

Hence, if $p$ is any positive integer, and if $n \geqq m$,

$$| u_{n+p} - u_n | = |(u_{n+p} - l) - (u_n - l)| \leqq | u_{n+p} - l | + | u_n - l | < \epsilon.$$

It will now be shown that the condition is also sufficient.

Consider a sequence $(\epsilon_n)$ of positive elements which decrease monotonically to the limit zero, and let $m_r$ be the value of $m$ corresponding to $\epsilon_r$, where $r$ is any positive integer: then $| u_{n+p} - u_n | < \epsilon_r$ if $n \geqq m_r$. Let $P_n$, for every $n$, be the point whose abscissa is $u_n$, $R_1$ and $S_1$ the points whose abscissæ are $u_{m_1} - \epsilon_1$ and $u_{m_1} + \epsilon_1$ respectively. Then the segment $R_1S_1$ is of length $2\epsilon_1$, and, if $n \geqq m_1$,

$$| u_n - u_{m_1} | < \epsilon_1,$$

or

$$u_{m_1} - \epsilon_1 < u_n < u_{m_1} + \epsilon_1,$$

so that $P_n$ lies within $R_1S_1$.

Now let $R'_2 S'_2$ be the segment corresponding in the same manner to $\epsilon_2$. If any part of $R'_2 S'_2$ lies outside $R_1S_1$ (Fig. 2), let it be cut off, and let the remaining segment be denoted by $R_2S_2$. Then $R_2S_2$ lies within $R_1S_1$ and is of length $\leqq 2\epsilon_2$. Every $P_n$, for which $n \geqq m_2$ lies within $R_2S_2$.

Proceeding in this way, we obtain a sequence of segments $(R_nS_n)$, each lying within the preceding one, and such that the length $R_nS_n \to 0$ when $n \to \infty$. The abscissæ of the points $R_n$ form a sequence which is monotonic increasing and bounded above (since $R_n$ lies to the left of $S_1$). Hence it converges to a limit. Similarly, the sequence of abscissæ of the points $S_n$ converges to a limit; and these limits are equal, since their difference is less than $R_nS_n$, which tends to zero when $n \to \infty$. Denote this common limit by $l$. Then $l$ is the limit of the sequence $(u_n)$. For,

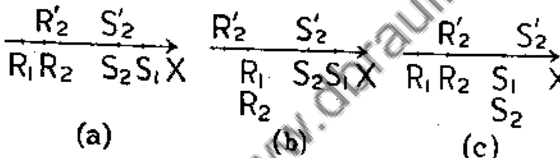


FIG. 2.

if $\epsilon$ be assigned, $r$ can be chosen so large that $2\epsilon_r < \epsilon$, and then, if $n \geqq m_r$,

$$|u_n - l| < R_rS_r \leqq 2\epsilon_r < \epsilon.$$

*Example* 6.—If $u_n = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$, show that the sequence $(u_n)$ is convergent.

Here

$$u_{n+p} - u_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2}$$
$$< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(n+p-1)(n+p)},$$

so that

$$u_{n+p} - u_n < \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{n+p-1}$$
$$- \frac{1}{n+1} - \frac{1}{n+2} - \ldots - \frac{1}{n+p},$$

or $$u_{n+p} - u_n < \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n} \leqq \frac{1}{m},$$

if $n \geqq m$.

But $m$ can always be chosen so large that $1/m < \epsilon$; then, if $n \geqq m$,

$$u_{n+p} - u_n < \epsilon.$$

## § 2. Convergence of Series

If $S_n$ denotes the sum of the first $n$ terms of the series

$$u_1 + u_2 + u_3 + \ldots,$$

the series converges if the sequence $(S_n)$ is convergent. It follows from the General Principle of Convergence that the series is convergent if, corresponding to any assigned positive quantity $\epsilon$, however small, an integer $m$ can be found such that, for $n \geqq m$,

$$| S_{n+p} - S_n | = | u_{n+1} + u_{n+2} + \ldots + u_{n+p} | < \epsilon,$$

where $p$ may be any positive integer.

The above condition may be written

$$| {}_pR_n | < \epsilon,$$

where ${}_pR_n \equiv S_{n+p} - S_n$. This quantity is called the *partial remainder after n terms*.

*Example* 1.—Show that, if the series $\Sigma u_n$ converges to U, the series $\Sigma k u_n$ converges to $k$U. [*Cf*. Example 1, § 1.]

*Example* 2.—Show that, if the series $\Sigma u_n$ and $\Sigma v_n$ converge to U and V respectively, the series $\Sigma(u_n + v_n)$ converges to (U + V).

Deduce that, if the series $\Sigma w_n$ converges to W, the series $\Sigma(u_n + v_n + w_n)$ converges to (U + V + W).

Since $u_{n+1} = {}_1R_n$, it follows that, if the series is convergent, $u_n$ must tend to zero as $n \to \infty$. This condition is necessary for convergence, but it is not sufficient. For instance, in the following example $u_n$ tends to zero, but the series is divergent.

*Example* 3.—[*The Harmonic Series.*] In the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots$$

let the terms after the first be combined in groups $v_1, v_2, v_3, v_4, \ldots$ containing 1, 2, $2^2$, $2^3$, . . . terms respectively, and let $v_0 = 1$. Then

$$v_1 = \tfrac{1}{2},\ v_2 = \tfrac{1}{3} + \tfrac{1}{4} > \tfrac{2}{4} = \tfrac{1}{2},\ v_3 = \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{4}{8} = \tfrac{1}{2},$$

and so on. The number of terms in the first $n$ groups $v_0, v_1, \ldots, v_{n-1}$ is

$$1 + 1 + 2 + 2^2 + \ldots + 2^{n-2} = 2^{n-1},$$

and therefore

$$v_n = \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \ldots + \frac{1}{2^{n-1}+2^{n-1}} > \frac{2^{n-1}}{2^n} = \tfrac{1}{2}.$$

Hence, if $p = 2^n$, $S_p \geqq 1 + \tfrac{1}{2}n$; and, consequently, when $n \to \infty$, $S_p \to \infty$. Thus the series is divergent.

*Example* 4.—Show that the series whose $n$th term is

$$\sqrt{(n^2 + n + 1)} - \sqrt{(n^2 - n + 1)}$$

is divergent.

## § 3. Series of positive Terms

In this section three important tests for the convergence of series whose terms are all positive will be given.

*The Comparison Test.*—Let $\Sigma u_n$ and $\Sigma a_n$ be series of positive terms. Then, if the series $\Sigma a_n$ is convergent, and if, for all values of $n$, $u_n \leqq a_n$, the series $\Sigma u_n$ is also convergent; while if the series $\Sigma a_n$ is divergent, and if, for all values of $n$, $u_n \geqq a_n$, the series $\Sigma u_n$ is also divergent.

Case I.—$\Sigma a_n$ convergent and $u_n \leqq a_n$. Let $U_n$ and $A_n$ be the sums to $n$ terms of $\Sigma u_n$ and $\Sigma a_n$ respectively, and let A be the sum to infinity of the latter series. Then, since all the terms are positive,

$$U_n \leqq A_n \leqq A,$$

and

$$U_{n+1} \geqq U_n.$$

The sequence $(U_n)$ is therefore monotonic increasing and bounded above. Hence, by Theorem I of § 1, it converges to a limit U, where $U \leqq A$.

CASE II.—$\Sigma a_n$ divergent and $u_n \geqq a_n$. For all values of $n$, $U_n \geqq A_n$: but, when $n$ tends to infinity, $A_n$ tends to infinity; hence $U_n$ also tends to infinity.

COROLLARY I.—In Case I, if $u_n \leqq ka_n$, where $k$ is positive and independent of $n$, $\Sigma u_n$ is convergent. For $\Sigma ka_n$ is convergent. In Case II, if $u_n \geqq ka_n$, $\Sigma u_n$ is divergent.

*Example* 1.—Show that the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent.

[Since $1/n^2 < 1/\{(n-1)n\}$, $n = 2, 3, 4, \ldots$, this follows by comparison with the convergent series $\sum_{n=2}^{\infty} 1/\{(n-1)n\}$, (*cf.* Ch. XVI, § 3, Example 1): or see § 1, Example 6.]

*Example* 2.—Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$, (i) converges if $s > 1$, (ii) diverges if $s \leqq 1$.

(i) If $s \geqq 2$, the result follows from Example 1 by the Comparison Test. If $1 < s < 2$, group the terms as follows:

$$(1) + \left(\frac{1}{2^s} + \frac{1}{3^s}\right) + \left(\frac{1}{4^s} + \ldots + \frac{1}{7^s}\right) + \left(\frac{1}{8^s} + \ldots + \frac{1}{15^s}\right) + \ldots .$$

Then
$$\left(\frac{1}{2^s} + \frac{1}{3^s}\right) < \frac{2}{2^s} = \frac{1}{2^{s-1}},$$
$$\left(\frac{1}{4^s} + \ldots + \frac{1}{7^s}\right) < \frac{4}{4^s} = \frac{1}{4^{s-1}},$$
and so on.

Thus, if $S_n$ denotes the sum to $n$ terms of the series, and if

$$p = 1 + 2 + 4 + \ldots + 2^{m-1} = 2^m - 1,$$
$$S_p < \left\{1 - \frac{1}{(2^{s-1})^m}\right\} \Big/ \left\{1 - \frac{1}{2^{s-1}}\right\} < 1 \Big/ \left\{1 - \frac{1}{2^{s-1}}\right\}.$$

Now $m$ can always be chosen so that $p \geqq n$. Thus $S_n < 1/\{1 - 1/2^{s-1}\}$. The sequence $(S_n)$ is therefore bounded above. But it is monotonic increasing. Therefore it is convergent.

(ii) The result can be derived from the harmonic series (§ 2, Example 3) by applying the Comparison Test.

Corollary II.—If $\Sigma u_n$ and $\Sigma a_n$ are series of positive terms, and if $u_n/a_n$ tends to $c$, a (positive) non-zero constant, when $n$ tends to infinity, $\Sigma u_n$ is convergent or divergent according as $\Sigma a_n$ is convergent or divergent.

Let $\epsilon$ be a positive number less than $c$; then a positive integer $m$ can be found such that, if $n \geqq m$,

$$c - \epsilon < u_n/a_n < c + \epsilon.$$

Hence, if $n \geqq m$,

$$u_n < (c + \epsilon)a_n \text{ and } u_n > (c - \epsilon)a_n.$$

It follows,* by Corollary I, that if $\Sigma a_n$ is convergent, $\Sigma u_n$ is convergent;† while, if $\Sigma a_n$ is divergent, $\Sigma u_n$ is divergent.

*Example* 3.—Show that the series $\Sigma\{\sqrt{(n+1)}/(n^2+n+1)\}$ is convergent, and that the series $\Sigma\{1/\sqrt{(n^2+n+1)}\}$ is divergent. [Compare them with the series $\Sigma(1/n^{3/2})$ and $\Sigma(1/n)$ respectively.]

*The Ratio Test.*—The series of positive terms $\Sigma u_n$ is convergent if, for all values of $n$ greater than, or equal to, some value $m$, $u_{n+1}/u_n \leqq r < 1$, where $r$ is independent of $n$; and is divergent if, for all values of $n$ greater than or equal to some value $m$, $u_{n+1}/u_n \geqq r > 1$, where $r$ is independent of $n$.

* The addition or subtraction of a finite number of terms to or from a series does not alter the convergence or divergence of the series: it merely alters the value of the sum of the series, if it is convergent. Thus, for the convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$, it is sufficient to prove that the series $\sum_{n=m}^{\infty} u_n$ is convergent or divergent.

† In the case of convergence $c$ may be zero; for then, if $\epsilon$ is any positive number, $m$ can be found so that $u_n/a_n < \epsilon$, and consequently $u_n < \epsilon a_n$, $n \geqq m$.

CASE I.—$u_{n+1}/u_n < r < 1$, where $n \geqq m$. Then

$$\frac{u_n}{u_m} = \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \quad \cdots \quad \frac{u_{m+1}}{u_m} \leqq r^{n-m}.$$

Hence

$$u_n \leqq u_m r^{n-m}.$$

But, since $r < 1$, the series $\sum_{n=m}^{\infty} u_m r^{n-m}$ is convergent (Ch. XVI, § 4, Example 2). Hence, by the Comparison Test, the series $\sum_{n=m}^{\infty} u_n$ is convergent; and, consequently, the series $\sum_{n=1}^{\infty} u_n$ is also convergent.

CASE II.—$u_{n+1}/u_n \geqq r > 1$, where $n \geqq m$. Then $u_n$ is $\geqq u_m r^{n-m}$. But, since $r > 1$, $r^{n-m}$ tends * to infinity when $n$ tends to infinity. Hence $u_n$ tends to infinity when $n$ tends to infinity, and consequently the series is divergent.

COROLLARY.—If, when $n$ tends to infinity, $u_{n+1}/u_n$ tends to a definite (positive) limit $r$, the series converges if $r < 1$, diverges if $r > 1$.

CASE I.—$0 \leqq r < 1$. Let $\rho$ be a number between $r$ and 1. Then $m$ can be chosen so large that, if $n \geqq m$,

$$\left| \frac{u_{n+1}}{u_n} - r \right| < \rho - r,$$

and therefore

$$\frac{u_{n+1}}{u_n} < r + (\rho - r) = \rho.$$

But $\rho < 1$; hence the series is convergent.

* If $r > 1$, let $r = 1 + h$, where $h > 0$; then, if $n$ is a positive integer, $r^n > 1 + nh$. But $1 + nh$ tends to infinity when $n$ tends to infinity. Therefore $r^n$ tends to infinity when $n$ tends to infinity.

CASE II.—$r > 1$. Let $\rho$ be a number between 1 and $r$. Then $m$ can be chosen so large that, if $n \geqq m$,

$$\left|\frac{u_{n+1}}{u_n} - r\right| < r - \rho,$$

and therefore $\dfrac{u_{n+1}}{u_n} > r - (r - \rho) = \rho.$

But $\rho > 1$; hence the series is divergent.

*Note.*—If $r = 1$, other tests must be applied. (See, for example, § 4, Theorem.)

*Monotonic Functions.*—If, in the interval between two given values of $x$ the function $f(x)$ does not decrease as $x$ increases, it is said to be *monotonic increasing* in the interval; if $f(x)$ does not increase as $x$ increases, the function is *monotonic decreasing* in the interval.

THEOREM.—If $f(x)$ is positive and monotonic decreasing for $x \geqq 1$, and if

$$u_n = f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)dx,$$

where $n = 1, 2, 3, \ldots$ the sequence $(u_n)$ converges to a limit $l$, where $0 \leqq l \leqq f(1)$.*

If $r \leqq x \leqq r + 1$, where $r$ is a positive integer,

$$f(r) \geqq f(x) \geqq f(r + 1),$$

and therefore

$$\int_r^{r+1} f(r)dx \geqq \int_r^{r+1} f(x)dx \geqq \int_r^{r+1} f(r + 1)dx,$$

or

$$f(r) \geqq \int_r^{r+1} f(x)dx \geqq f(r + 1).$$

Thus

$$u_n = f(1) - \sum_{r=1}^{n-1}\left\{\int_r^{r+1} f(x)dx - f(r + 1)\right\},$$

so that $$u_n \leqq f(1) \quad \text{and} \quad u_{n+1} \leqq u_n.$$

* If $f(x)$ decreases *continuously* for $x \geqq 1$, $0 < l < f(1)$.

Also

$$u_n = \sum_{r=1}^{n-1} \left\{ f(r) - \int_r^{r+1} f(x)dx \right\} + f(n) \geqq 0.$$

Therefore the sequence $(u_n)$ is monotonic decreasing and is bounded below. The result follows from § 1, Theorem II.

Euler's Constant.—In the above theorem let $f(x) = 1/x$; then

$$u_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n.$$

When $n$ tends to infinity, $u_n$ tends to a limit between 0 and 1. This limit is known as Euler's Constant, and is denoted by $\gamma$ or C. Its value is

$$0{\cdot}577\ 215\ 664\ 90\ \ldots$$

*Maclaurin's Integral Test.*—If the function $f(x)$ is positive and monotonic decreasing for $x \geqq 1$, the series $\sum_{r=1}^{\infty} f(r)$ converges or diverges according as the integral $\int_1^{\infty} f(x)dx$ converges or diverges.

Let $S_n$ be the sum of the first $n$ terms of the series. Then, in the theorem above,

$$S_n = \int_1^n f(x)dx + u_n.$$

Now, when $n$ tends to infinity, $u_n$ tends to $l$. Hence $S_n$ tends to a definite limit or to infinity according as the integral tends to a definite limit or to infinity.

*Example* 4.—Show that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent.

*Example* 5.—If $s \neq 1$, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges or diverges according as $s > 1$ or $s < 1$.

*Example* 6.—If $x$ is positive, show that

$$\underset{x\to 0}{\mathcal{L}}\left\{\sum_{n=1}^{\infty}\frac{x}{1+n^2x^2}\right\}=\frac{\pi}{2}.$$

[If $x>0$, $\sum_{n=1}^{\infty}\frac{x}{1+n^2x^2}=\int_1^{\infty}\frac{x\,d\xi}{1+\xi^2x^2}+\phi(x)$

$$=\tfrac{1}{2}\pi-\tan^{-1}x+\phi(x),\ \text{where } 0\leqq\phi(x)\leqq\frac{x}{1+x^2}.]$$

*Example* 7.—Show that the series

$$\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$$

converges or diverges according as $p>1$ or $p\leqq 1$.

[In the Integral Test put $f(x)=1/\{x(\log x)^p\}$, and take 2 as the lower limit. Then, if $p=1$,

$$\int_2^n\frac{dx}{x\log x}=\log(\log n)-\log(\log 2);$$

while, if $p\neq 1$,

$$\int_2^n\frac{dx}{x(\log x)^p}=\frac{(\log n)^{1-p}-(\log 2)^{1-p}}{1-p}.]$$

## § 4. Absolute Convergence

The convergence of series whose terms are not all positive will now be considered.

*Absolute Convergence.*—If the series of moduli $\Sigma|u_n|$ is convergent, the series $\Sigma u_n$ is itself convergent, and is said to be absolutely convergent.

The convergence of $\Sigma u_n$ follows from the inequality (Ch. XIII, § 7, Theorem I).

$$|u_{n+1}+u_{n+2}+\ldots+u_{n+p}|$$
$$\leqq|u_{n+1}|+|u_{n+2}|+\ldots+|u_{n+p}|.$$

For, since $\Sigma|u_n|$ is convergent, $m$ can be chosen so large that, for $n\geqq m$, the R.H.S. $<\epsilon$; hence the L.H.S. is also less than $\epsilon$.

*Example* 1.—Show that the series $\sum \frac{\cos n\theta}{n^2}$, $\sum \frac{\sin n\theta}{n^2}$ are absolutely convergent for all values of $\theta$.

Since

$$\left|\frac{\cos n\theta}{n^2}\right| \leqq \frac{1}{n^2}, \quad \left|\frac{\sin n\theta}{n^2}\right| \leqq \frac{1}{n^2},$$

the results follow, by means of the Comparison Test, from § 3, Example 1.

*Example* 2.—Show that the series $\Sigma r^n \cos n\theta$, $\Sigma r^n \sin n\theta$ are absolutely convergent if $|r| < 1$.

*Example* 3.—If $|r| < 1$, show that

(i) $$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} = 1 + r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \ldots,$$

(ii) $$\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} = r\sin\theta + r^2\sin 2\theta + r^3\sin 3\theta + \ldots,$$

(iii) $$\frac{1 - r^2}{1 - 2r\cos\theta + r^2} = 1 + 2r\cos\theta + 2r^2\cos 2\theta + 2r^3\cos 3\theta + \ldots,$$

(iv) $$\frac{\cos\theta - r}{1 - 2r\cos\theta + r^2} = \cos\theta + r\cos 2\theta + r^2\cos 3\theta + \ldots,$$

(v) $$\frac{(1 - r)\cos\theta}{1 - 2r\cos 2\theta + r^2} = \cos\theta + r\cos 3\theta + r^2\cos 5\theta + \ldots.$$

[Multiply the series on the right-hand sides of the equations by $1 - 2r\cos\theta + r^2$, or, for (v), by $1 - 2r\cos 2\theta + r^2$, and add the coefficients of the different powers of $r$. See also Ch. XV, § 9, Examples 5, 6, and Ch. XX, § 5, Example 2.]

*Example* 4.—If $\alpha + p > 0$, where $p$ is a positive integer, show that the series

$$\sum_{n=p}^{\infty} \left\{\log\left(1 + \frac{\alpha}{n}\right) - \frac{\alpha}{n}\right\}$$

is absolutely convergent.

Choose $m$, a positive integer ($\geqq p$), so large that, for $n \geqq m$, $|\alpha/n| < 1$. Then, if $n \geqq m$,

$$\log\left(1 + \frac{\alpha}{n}\right) = \frac{\alpha}{n} - \frac{1}{2}\frac{\alpha^2}{n^2} + \frac{1}{3}\frac{\alpha^3}{n^3} - \ldots;$$

and therefore

$$\left|\log\left(1+\frac{\alpha}{n}\right)-\frac{\alpha}{n}\right| \leqq \left|\frac{\alpha^2}{n^2}\right| + \left|\frac{\alpha^3}{n^3}\right| + \left|\frac{\alpha^4}{n^4}\right| + \ldots,$$

$$\leqq \frac{|\alpha|^2}{n^2}\left\{1+\left|\frac{\alpha}{m}\right|+\left|\frac{\alpha}{m}\right|^2+\ldots\right\},$$

or

$$\left|\log\left(1+\frac{\alpha}{n}\right)-\frac{\alpha}{n}\right| \leqq \frac{|\alpha|^2}{1-|\alpha/m|}\cdot\frac{1}{n^2},$$

where $n \geqq m$.

Now the series $\sum \frac{1}{n^2}$ is convergent. Hence, by the Comparison Theorem, the series

$$\sum_{n=m}^{\infty}\left\{\log\left(1+\frac{\alpha}{n}\right)-\frac{\alpha}{n}\right\}$$

is absolutely convergent. It follows that the given series is absolutely convergent.

*Power Series.*—A series of the type $\Sigma a_n x^n$ is called a power series. If, when $n$ tends to infinity, $|a_n/a_{n+1}|$ tends to R, it follows from the Ratio Test that the series is absolutely convergent if

$$\mathop{\mathcal{L}}_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \frac{|x|}{\mathrm{R}} < 1;$$

*i.e.*, if

$$-\mathrm{R} < x < \mathrm{R}.$$

The distance R is called the *radius of convergence*, and the interval from $-$ R to R, denoted by ($-$ R, R), is the *interval of convergence.*

*Open and Closed Intervals.*—The numbers $x$ which satisfy the inequalities $a \leqq x \leqq b$ form the *closed* interval $(a, b)$, while the numbers $x$ for which $a < x < b$ form the *open* interval $(a, b)$. If $a < x \leqq b$ the interval $(a, b)$ is open at $a$ and closed at $b$, while if $a \leqq x < b$ the interval is closed at $a$ and open at $b$. The interval ($-$ R, R) above is an open interval.

*Example* 5.—Show that the series $\Sigma x^n/n$ is absolutely convergent in the open interval $(-1, 1)$.

*Example* 6.—Show that the series $\Sigma x^n/n^2$ is absolutely convergent in the closed interval $(-1, 1)$.

*Example* 7.—Show that the series $\Sigma x^n/n!$ is absolutely convergent for all values of $x$.

*The Hypergeometric Function.*—This function is defined by the series

$$F(\alpha, \beta\,;\ \gamma\,;\ x) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 2\,!}x^2 + \ldots,$$

the $(n+1)$th term being

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)\,.\,n\,!}x^n,$$

provided that $\gamma$ is not zero or a negative integer.

Thus

$$\frac{u_{n+2}}{u_{n+1}} = \frac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)}x = \frac{\left(1+\dfrac{\alpha}{n}\right)\left(1+\dfrac{\beta}{n}\right)}{\left(1+\dfrac{\gamma}{n}\right)\left(1+\dfrac{1}{n}\right)}x,$$

and this tends to $x$ when $n$ tends to infinity. Hence the series converges absolutely if $|x| < 1$.

The convergence for the cases $x = \pm 1$ can be investigated by means of the following theorem:

THEOREM.—If, for all positive integral values of $n$,

$$u_n = \frac{(\alpha+1)(\alpha+2)\ \ldots\ (\alpha+n)}{(\beta+1)(\beta+2)\ \ldots\ (\beta+n)},$$

where $\beta$ is not a negative integer, then

$$|\,u_n\,| < \frac{A}{n^{\beta-\alpha}},$$

A being a (positive) constant independent of $n$.

This theorem follows from the following lemmas:

LEMMA I.—If $\alpha$ is not a negative integer, and if, for all positive integral values of $n$,

$$v_n = \prod_{r=1}^{n}\left\{\left(1+\frac{\alpha}{r}\right)e^{-\frac{\alpha}{r}}\right\},$$

the sequence $(v_n)$ converges to a definite non-zero limit.

For, if $p$ is a positive integer such that $\alpha + p > 0$, and if

$$w_n = \prod_{r=p}^{n}\left\{\left(1+\frac{\alpha}{r}\right)e^{-\alpha/r}\right\},\ n = p, p+1, p+2, \ldots,$$

then

$$w_n = e^{s_n},$$

where

$$s_n = \sum_{r=p}^{n}\left\{\log\left(1+\frac{\alpha}{r}\right) - \frac{\alpha}{r}\right\}.$$

But by Example 4, $s_n$ tends to a definite limit when $n$ tends to infinity. Hence $w_n$ tends to a definite non-zero limit when $n$ tends to infinity. Now

$$v_n = w_n \times \text{a non-zero constant}.$$

Therefore $v_n$ tends to a definite non-zero limit when $n$ tends to infinity.

LEMMA II.—If $\alpha$ is not a negative integer, and if

$$v_n = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!\,n^{\alpha}},\ n = 1, 2, 3, \ldots,$$

the sequence $(v_n)$ converges to a definite non-zero limit.

For

$$v_n = \prod_{r=1}^{n}\left\{\left(1+\frac{\alpha}{r}\right)e^{-\frac{\alpha}{r}}\right\}.\,e^{\alpha\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right)}$$

and the result follows from Lemma I and the property of Euler's Constant established in the previous section.

LEMMA III.—If $\alpha$ and $\beta$ are not negative integers, and if

$$v_n = \frac{(\alpha+1)(\alpha+2)\ .\ .\ .\ (\alpha+n)}{(\beta+1)(\beta+2)\ .\ .\ .\ (\beta+n)} n^{\beta-\alpha},\ \ n = 1, 2, 3\ .\ .\ .\ ,$$

the sequence $(v_n)$ converges to a definite non-zero limit.

For

$$v_n = \frac{(\alpha+1)(\alpha+2)\ .\ .\ .\ (\alpha+n)}{n\,!\ n^{\alpha}} \div \frac{(\beta+1)(\beta+2)\ .\ .\ .\ (\beta+n)}{n\,!\ n^{\beta}},$$

and the result follows from Lemma II.

As the sequence $(v_n)$ in Lemma III is convergent, it is bounded (see § 1, Note), that is, there is a positive number A, such that

$$|\,v_n\,| < A,\ n = 1, 2, 3\ .\ .\ .$$

From this the theorem follows.

For the hypergeometric function $F(\alpha, \beta\,;\ \gamma\,;\ x)$ it follows that, if $\gamma$ is not a negative integer,

$$|\,u_{n+1}| < \frac{A}{n^{\gamma-\alpha-\beta+1}}\,|\,x\,|^n.$$

Thus, if $|\,x\,| = 1$, the series converges absolutely if $\gamma - \alpha - \beta > 0$. [See § 3, Example 5.]

*Example* 8.—Prove that, when $n$ tends to infinity, the product

$$\prod_{r=1}^{n} \left\{\frac{(r+x)\,(r-y)}{(r-x)\,(r+y)}\right\},$$

where $x$ and $y$ are not integers, tends to one of the values $0, 1, \pm\infty$.

*Example* 9.—Prove that the series

$$\frac{x}{x+3} + \frac{x(x+2)}{(x+3)(x+5)} + \frac{x(x+2)\,(x+4)}{(x+3)(x+5)(x+7)} + \ .\ .\ .$$

is absolutely convergent unless $x$ is an odd negative integer other than $-1$, and verify that its sum when convergent is $x$.

*Example* 10.—Show that the series

$$1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots$$

is absolutely convergent if $b > a + 1$, unless $b$ is zero or a negative integer, and verify that its sum when convergent is $(b-1)/(b-a-1)$.

*Example* 11.—If in the series $\Sigma a_n$ of positive terms the ratio $a_{n+1}/a_n$ can always be expressed in the form

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{k_n}{n^p},$$

where $p > 1$ and $|k_n| \leqq K$, K being a (positive) constant independent of $n$, prove that the series is convergent if $\mu > 1$, divergent if $\mu \leqq 1$.

Let $m$ be a positive integer so large that, if

$$\rho_m = \frac{|\mu|}{m} + \frac{K}{m^p},$$

$\rho_m < 1$. Then, if

$$u_r = \frac{\mu}{r} - \frac{k_r}{r^p}, \quad r = m, m+1, m+2, \dots,$$

$$\left|\log\left(1 - \frac{\mu}{r} + \frac{k_r}{r^p}\right) + \frac{\mu}{r}\right| = \left|\frac{k_r}{r^p} - \frac{1}{2}u_r^2 - \frac{1}{3}u_r^3 - \dots\right|$$

$$\leqq \frac{K}{r^p} + \frac{1}{2}|u_r|^2 \times (1 + \rho_m + \rho_m^2 + \dots)$$

$$\leqq \frac{K}{r^p} + \frac{m^2\rho_m^2}{2(1-\rho_m)} \cdot \frac{1}{r^2}.$$

Hence, by the Comparison Test,

$$\sum_{r=m}^{\infty} \{\log(1 - u_r) + \mu/r\}$$

is absolutely convergent.

It follows that, if

$$w'_n = \prod_{r=m}^{n} \left\{(1-u_r)e^{\frac{\mu}{r}}\right\},$$

$w'_n$ tends to a definite non-zero limit when $n$ tends to infinity, and that this is also true of

$$w_n = \prod_{r=1}^{n} \left\{(1 - u_r)e^{\frac{\mu}{r}}\right\},$$

provided that none of the factors vanishes.

Next, let

$$v_n = \prod_{r=1}^{n} (1 - u_r) \times n^{\mu}.$$

Then $$v_n = w_n \times e^{-\mu\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right)},$$

and therefore $v_n$ tends to a definite non-zero limit when $n$ tends to infinity, provided that none of the factors vanishes.

Now

$$a_{n+1} = a_1 \prod_{r=1}^{n} (1 - u_r).$$

Hence, by the Comparison Test, Corollary II, $\Sigma a_n$ converges or diverges according as $\Sigma(1/n^{\mu})$ converges or diverges.

*Example* 12.—Show that the series

$$\sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1^2)(\alpha + 2^2) \ldots \{\alpha + (n-1)^2\}}{\beta(\beta + 1^2)(\beta + 2^2) \ldots \{\beta + (n-1)^2\}},$$

where $\alpha > 0$, $\beta > 0$, is divergent.



## § 5. Conditional Convergence

A series which is convergent, but is not absolutely convergent, is said to be conditionally convergent.

In testing for conditional convergence the following theorem is often found useful.

*Abel's Inequality*.—If $(c_n)$ is a monotonic decreasing sequence whose elements are all positive, and if

$$h \leqq u_1 + u_2 + \ldots + u_r \leqq \mathrm{H},$$

where $h$ and H are independent of $r$ and $r = 1, 2, 3, \ldots n$, then

$$c_1 h \leqq c_1 u_1 + c_2 u_2 + \ldots + c_n u_n \leqq c_1 \mathrm{H}.$$

Let $$S = c_1u_1 + c_2u_2 + \ldots + c_nu_n,$$
and $$s_r = u_1 + u_2 + \ldots + u_r,$$
where $r = 1, 2, \ldots, n$. Then

$$\begin{aligned} S &= c_1s_1 + c_2(s_2 - s_1) + c_3(s_3 - s_2) + \ldots + c_n(s_n - s_{n-1}) \\ &= (c_1 - c_2)s_1 + (c_2 - c_3)s_2 + \ldots + (c_{n-1} - c_n)s_{n-1} + c_ns_n. \end{aligned}$$

Now the coefficients of $s_1, s_2, \ldots, s_n$ are all positive (or zero), and each of the quantities $s_r$ lies between $h$ and H. Hence

$$S \leqq \{(c_1 - c_2) + (c_2 - c_3) + \ldots + (c_{n-1} - c_n) + c_n\}H = c_1H,$$
and
$$S \geqq \{(c_1 - c_2) + (c_2 - c_3) + \ldots + (c_{n-1} - c_n) + c_n\}h = c_1h.$$

Therefore $$c_1h \leqq S \leqq c_1H.$$

*Example* 1.—Prove that the series $\sum \frac{\cos n\theta}{n}$ is convergent if $\theta \neq 2m\pi$, where $m$ is any integer.

From Ch. VII, (9) we have

$$\cos(n+1)\theta + \cos(n+2)\theta + \ldots + \cos(n+p)\theta = \cos\left(n + \frac{p+1}{2}\right)\theta \frac{\sin \frac{1}{2}p\theta}{\sin \frac{1}{2}\theta},$$

provided that $\frac{1}{2}\theta \neq m\pi$.

But

$$-1 \leqq \cos\left(n + \frac{p+1}{2}\right)\theta \sin \frac{1}{2}p\theta \leqq 1.$$

Hence, if $\theta \neq 2m\pi$,

$$-\frac{1}{|\sin \frac{1}{2}\theta|} \leqq \cos(n+1)\theta + \cos(n+2)\theta + \ldots + \cos(n+p)\theta \leqq \frac{1}{|\sin \frac{1}{2}\theta|}.$$

Thus, by Abel's Inequality, if $\theta \neq 2m\pi$,

$$-\frac{1}{(n+1)|\sin \frac{1}{2}\theta|} \leqq \sum_{r=n+1}^{n+p} \frac{\cos r\theta}{r} \leqq \frac{1}{(n+1)|\sin \frac{1}{2}\theta|},$$

or

$$\left|\sum_{r=n+1}^{n+p} \frac{\cos r\theta}{r}\right| \leqq \frac{1}{(n+1)|\sin \frac{1}{2}\theta|}.$$

But, by taking $n$ large enough, the quantity on the right can be made as small as we please. Hence the series converges if $\theta \neq 2m\pi$.

*Example* 2.—Prove that the series $\sum \frac{\sin n\theta}{n}$ is convergent if $\theta \neq 2m\pi$. If $\theta = 2m\pi$ it vanishes identically.

*Example* 3.—Prove that the series $\sum \frac{\cos n\theta}{n^s}$, $\sum \frac{\sin n\theta}{n^s}$ are convergent for $0 < s \leqq 1$ if $\theta \neq 2m\pi$, and that they converge absolutely for all values of $\theta$ if $s > 1$.

*Alternating Series.*—If $(a_n)$, a monotonic decreasing sequence whose elements are all positive, converges to zero, the series

$$a_1 - a_2 + a_3 - a_4 + \ldots$$

is convergent. A series of this type is called an *alternating series.*

To prove that the series is convergent, apply Abel's Inequality to the sum

$$A = a_{n+1} - a_{n+2} + a_{n+3} - \ldots \pm a_{n+p},$$

with $c_r = a_{n+r}$ and $s_r = 1 - 1 + 1 - \ldots$ to $r$ terms.

Here $h = 0$, $H = 1$, so that

$$0 \leqq A \leqq a_{n+1}.$$

Now, when $n$ tends to infinity, $a_{n+1}$ tends to zero, and therefore A tends to zero. But for the alternating series

$$A = |S_{n+p} - S_n|.$$

Hence the series is convergent.

*Example* 4.—Show that the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$

is convergent.

*Example* 5.—Show that the series $\sum \frac{x^n}{n}$ converges for $-1 \leqq x < 1$.

*Example* 6.—Show that the series

$$2 - 1\tfrac{1}{2} + 1\tfrac{1}{3} - 1\tfrac{1}{4} + \ldots$$

is not convergent.

$$\left[u_n = \pm\left(1 + \frac{1}{n}\right), \text{ and does not } \to 0 \text{ when } n \to \infty.\right]$$

*Example* 7.—**The Hypergeometric Function.** Show that the series $F(\alpha, \beta; \gamma; -1)$ is conditionally convergent if

$$-1 < \gamma - \alpha - \beta \leqq 0.$$

Here

$$1 - \left|\frac{u_{n+2}}{u_{n+1}}\right| = 1 - \frac{(\alpha + n)(\beta + n)}{(\gamma + n)(1 + n)}$$
$$= \frac{n\left(\gamma - \alpha - \beta + 1 + \frac{\gamma - \alpha\beta}{n}\right)}{(\gamma + n)(1 + n)}.$$

Since $\gamma - \alpha - \beta + 1 > 0$, it follows that this is positive if $n$ is large enough. Hence, for $n$ large,

$$|u_{n+2}| < |u_{n+1}|.$$

Again

$$|u_{n+1}| < \frac{A}{n^{\gamma-\alpha-\beta+1}},$$

and therefore $u_{n+1} \to 0$ when $n \to \infty$. Thus the series is ultimately alternating, and consequently convergent.

*Derangement of Series.*—If the order of the terms in a series is altered, the series is said to be deranged. If the series is absolutely convergent, it can be proved (see Dirichlet's Theorem below) that the deranged series converges to the same sum as before. On the other hand, if the series only converges conditionally, the deranged series may converge to a different sum, or may be divergent (see Examples 8, 9 below). For this reason absolutely convergent series are said to converge *unconditionally*, while series which converge, but not absolutely, are said to be *conditionally convergent*.

*Dirichlet's Theorem.*—The sum of an absolutely convergent series is not altered if the series is deranged.

Case I.—Let $\Sigma a_n$ be a convergent series of positive terms whose sum is $s$, $\Sigma b_n$ the deranged series, $s_n$ and $\sigma_n$ the sums to $n$ terms of $\Sigma a_n$ and $\Sigma b_n$ respectively.

Since every term in $\Sigma b_n$ occurs in $\Sigma a_n$, it is possible to take $n$ so large that every term in $\sigma_m$ occurs in $s_n$. Then

$$\sigma_m \leqq s_n \leqq s.$$

Thus the sequence $(\sigma_m)$ is bounded above. But it is monotonic increasing; hence it converges to a limit $\sigma$, where $\sigma \leqq s$.

Again, since $\Sigma a_n$ is a derangement of $\Sigma b_n$, $s \leqq \sigma$. It follows that $\sigma = s$.

CASE II.—Let there be an infinite number of positive and an infinite number of negative terms in the absolutely convergent series $\Sigma a_n$.

Let $P_\mu$ and $-Q_\nu$ be the sums of the positive and negative terms respectively in $s_n$; so that $\mu + \nu = n$ and $\mu$ and $\nu$ tend to infinity when $n$ tends to infinity. Then, if

$$\rho_n = |a_1| + |a_2| + \ldots + |a_n|,$$
$$P_\mu + Q_\nu = \rho_n, \quad P_\mu - Q_\nu = s_n,$$

and therefore

$$P_\mu = \tfrac{1}{2}(\rho_n + s_n), \quad Q_\nu = \tfrac{1}{2}(\rho_n - s_n).$$

Now, when $n$ tends to infinity, $\rho_n$ and $s_n$ tend to limits $\rho$ and $s$ respectively. Hence $P_\mu$ and $Q_\nu$ tend to limits P and Q respectively, where

$$P = \tfrac{1}{2}(\rho + s), \; Q = \tfrac{1}{2}(\rho - s).$$

But the series whose sums are P and Q are convergent series of positive terms, and their sums are therefore not altered by derangement. Hence $s$, which is equal to $P - Q$, is not altered by derangement.

*Example* 8.—The conditionally convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots$$

converges to the sum $\log 2$ {Ch. XVII, (31)}. Show that the deranged series

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \ldots$$

converges to the sum $\tfrac{3}{2} \log 2$.

Let $s_n$ and $\sigma_n$ be the sums to $n$ terms of the first and second series respectively. Then

$$s_{4n} = \sum_1^n \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right),$$

and $$\sigma_{3n} = \sum_1^n \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right).$$

Hence $$\sigma_{3n} - s_{4n} = \frac{1}{2} \sum_1^n \left( \frac{1}{2n-1} - \frac{1}{2n} \right) = \frac{1}{2} s_{2n};$$

so that

$$\sigma_{3n} = s_{4n} + \tfrac{1}{2} s_{2n}.$$

Thus, when $n$ tends to infinity, $\sigma_{3n}$ tends to $\frac{3}{2}\log 2$.

For an alternative proof see Examples XVIII, 3.

*Example* 9.—Show that the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{4} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \tfrac{1}{13} + \tfrac{1}{15} + \tfrac{1}{17} + \tfrac{1}{19} - \tfrac{1}{8} + \ldots$$

is divergent.

If $N = \frac{1}{2}n(n+1)$, the sum of the first $N + n$ terms of the series is

$$\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2N} - \log(2N)\right\}$$
$$- \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - \log N\right)$$
$$- \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) + \frac{1}{2}\log(2n+2).$$

When $n$ tends to infinity, each of the expressions in the brackets tends to Euler's Constant, while the last term tends to infinity.

## § 6. Multiplication of Series

Let the two series

$$u_0 + u_1 + u_2 + \ldots, \quad v_0 + v_1 + v_2 + \ldots$$

be absolutely convergent, their sums being U and V respectively. Then, if

$$w_n = u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \ldots + u_n v_0,$$

where $n = 0, 1, 2, \ldots$, the series

$$w_0 + w_1 + w_2 + w_3 + \ldots$$

converges absolutely to the sum UV.

Let $U_n$, $V_n$, $W_n$ be the sums to $n$ terms of the three series.

Case I.—Let all the $u$'s and $v$'s be positive.

Any term $u_pv_q$ in $W_n$ is included among the terms in the product $U_nV_n$, but only those terms $u_pv_q$ appear in $W_n$ for which $p$ and $q$ satisfy the inequality $p + q \leqq n - 1$. Therefore

$$W_n \leqq U_nV_n \leqq UV.$$

Thus the sequence $(W_n)$ is bounded above. But it is monotonic increasing; hence it converges to a limit W, where $W \leq UV$.

Again, the sum $W_{2n-1}$ contains all the terms $u_pv_q$ for which $p + q \leqq 2n - 2$; therefore

$$W_{2n-1} \geqq U_nV_n.$$

In this inequality let $n$ tend to infinity, and get $W \geqq UV$. Hence, as $W \leq UV$,

$$W = UV.$$

Case II.—The $u$'s and $v$'s are not all positive.

Let the sums corresponding to $U_n$, $V_n$, $W_n$, when all the terms are replaced by their moduli, be $U'_n$, $V'_n$, $W'_n$ respectively. Then, as $U_nV_n$ contains all the terms $u_pv_q$ in $W_n$, and others as well,

$$| U_nV_n - W_n | \leqq U'_nV'_n - W'_n,$$

since all the terms $| u_p | . | v_q |$ remaining on the right after the subtraction are the moduli of the corresponding terms remaining on the left. Now, by Case I, $U'_nV'_n - W'_n$ tends to zero when $n$ tends to infinity; hence $U_nV_n - W_n$ also tends to zero when $n$ tends to infinity. But $U_nV_n$ tends to UV; therefore $W_n$ tends to UV.

*Example* 1.—Prove by multiplication that

$$\exp(x)\exp(y) = \exp(x + y).$$

Here $$u_{n+1} = \frac{x^n}{n!},\ v_{n+1} = \frac{y^n}{n!},$$

and therefore

$$w_{n+1} = \frac{x^n}{n!} \cdot 1 + \frac{x^{n-1}}{(n-1)!} \cdot \frac{y}{1!} + \frac{x^{n-2}}{(n-2)!} \frac{y^2}{2!} + \ldots + 1 \cdot \frac{y^n}{n!}$$
$$= \frac{1}{n!}\left\{x^n + \frac{n}{1} x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \ldots + y^n\right\}$$
$$= \frac{(x+y)^n}{n!}.$$

*Example* 2.—Prove by multiplication of series that

$$\text{(i)} \quad e^{r \cos\theta} \cos(r \sin\theta) = \sum_{n=0}^{\infty} \frac{r^n}{n!} \cos n\theta,$$

$$\text{(ii)} \quad e^{r \cos\theta} \sin(r \sin\theta) = \sum_{n=1}^{\infty} \frac{r^n}{n!} \sin n\theta.$$

[For theorems of greater generality on the multiplication of series the reader is referred to Bromwich's "Infinite Series," second edition, pages 91-94.]



## § 7. Double Series

If each of the series

$$\sum_{n=1}^{\infty} u_{m,n} = A_m, \quad m = 1, 2, 3, \ldots$$

is convergent, and if the series $\sum_{m=1}^{\infty} A_m$ is also convergent, its sum being S, the latter series may be written

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{m,n}.$$

It is then called a *repeated series*, and is said to converge to the sum S.

It is sometimes of importance to know if the repeated series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{m,n},$$

in which the order of summation has been changed, is convergent, and if it has the same sum S. The conditions for this stated in the following theorem are *sufficient*, though not always *necessary*.

THEOREM.—If all the series

$$A_m = \sum_{n=1}^{\infty} u_{m,\,n}\,, \quad m = 1, 2, 3, \ldots$$

are absolutely convergent, and if the series $\sum\limits_{m=1}^{\infty} A'_m$, where

$$A'_m = \sum_{n=1}^{\infty} |\, u_{m,\,n} \,|\,, \quad m = 1, 2, 3, \ldots$$

is convergent, then the series

$$B_n = \sum_{m=1}^{\infty} u_{m,\,n}\,, \quad n = 1, 2, 3, \ldots, \quad \text{and} \sum_{u=1}^{\infty} B_n$$

are all absolutely convergent, and

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{m,\,n} = \sum_{n=1}^{\infty} B_n = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{m,\,n} = \sum_{m=1}^{\infty} A_m = S.$$

The terms of the double series may be arrayed as follows, with the sums opposite the respective rows and columns:

$$\begin{array}{cccccc|c}
u_{1,\,1}, & u_{1,\,2}, & u_{1,\,3}, & \ldots, & u_{1,\,n}, & \ldots & A_1 \\
u_{2,\,1}, & u_{2,\,2}, & u_{2,\,3}, & \ldots, & u_{2,\,n}, & \ldots & A_2 \\
u_{3,\,1}, & u_{3,\,2}, & u_{3,\,3}, & \ldots, & u_{3,\,n}, & \ldots & A_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
u_{m,\,1}, & u_{m,\,2}, & u_{m,\,3}, & \ldots, & u_{m,\,n}, & \ldots & A_m \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
B_1, & B_2, & B_3, & \ldots, & B_n, & \ldots &
\end{array}$$

The first method of summation is then said to give summation by rows, the second method summation by columns.

CASE I.—Let all the terms $u_{m,n}$ be positive, so that $A'_m$ is identical with $A_m$. Then, since $u_{m,n} \leq A_m$, where $m$ is any positive integer, and since the series $\Sigma A_m$ is convergent, so also, by the comparison test, are the series

$$B_n = \sum_{m=1}^{\infty} u_{m,n}, \quad n = 1, 2, 3, \ldots .$$

Again, if

$$\Sigma_n = B_1 + B_2 + \ldots + B_n,$$

since each of the series $B_1, B_2, \ldots, B_n$ is convergent, so also is the series obtained by adding their corresponding terms. Let $\sigma_m$ be the $m$th term of this series, so that

$$\sigma_m = u_{m,1} + u_{m,2} + \ldots + u_{m,n},$$

being the sum of the first $n$ terms of the $m$th row.

Then $\sigma_m \leqq A_m$, $m = 1, 2, 3, \ldots$, and therefore

$$\Sigma_n \leqq \sum_{m=1}^{\infty} A_m = S.$$

Thus the sequence $(\Sigma_n)$ is bounded above, and it is monotonic increasing. Hence it converges to a limit $\Sigma$, where $\Sigma \leqq S$.

Conversely, it can now be shown that $S \leqq \Sigma$. Therefore

$$\Sigma = S.$$

CASE II.—Let the terms $u_{m,n}$ be not all positive. Since $|u_{m,n}| \leqq A'_m$ and $\Sigma A'_m$ is convergent, the series

$$B_n = \sum_{m=1}^{\infty} u_{m,n}, \quad n = 1, 2, 3, \ldots$$

are all absolutely convergent.

As before, let

$$\Sigma_n = B_1 + B_2 + \ldots + B_n,$$

and

$$\sigma_m = u_{m,1} + u_{m,2} + \ldots + u_{m,n},$$

so that the series $\Sigma\sigma_m$ converges to the sum $\Sigma_n$. Also let $S'$, $\sigma'_m$, $\Sigma'_n$ be the sums corresponding to $S$, $\sigma_m$, $\Sigma_n$ when each $u_{m,n}$ is replaced by its modulus. Then

$$S - \Sigma_n = \sum_{m=1}^{\infty} A_m - \sum_{m=1}^{\infty} \sigma_m = \sum_{m=1}^{\infty} (A_m - \sigma_m)$$
$$= \sum_{m=1}^{\infty} (u_{m,n+1} + u_{m,n+2} + \ldots).$$

Similarly,

$$S' - \Sigma'_n = \sum_{m=1}^{\infty} (|u_{m,n+1}| + |u_{m,n+2}| + \ldots).$$

Hence

$$|S - \Sigma_n| \leqq |S' - \Sigma'_n|.$$

But, by Case I, when $n$ tends to infinity, $S' - \Sigma'_n$ tends to zero. Therefore $S - \Sigma_n$ tends to zero. Thus the series $\Sigma B_n$ converges to the sum S.

*Substitution of a Power Series in a Power Series.*—The above theorem is particularly useful when a power series is to be substituted in a power series. Thus if, in the power series $\Sigma a_n y^n$, $y$ is replaced by the series $\Sigma b_n x^n$, and the resulting series rearranged in powers of $x$, the sum is unaltered provided that the series $\Sigma b_n x^n$ and $\Sigma a_n Y^n$, where

$$Y = \Sigma\, |b_n x^n|,$$

are absolutely convergent.

*Example.*—**The Binomial Theorem.**

Prove that, if $-1 < x < 1$,

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots,$$

where $(1+x)^m$ has its principal value (Ch. XVII, § 5); that is, $(1+x)^m = 1$ when $x = 0$.

We have

$$(1+x)^m = e^{m \log(1+x)} = e^y$$

$$= 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots,$$

where $$y = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right),$$

a series which converges absolutely if $|x| < 1$.

Let $$Y = |m| \left(|x| + \frac{|x^2|}{2} + \frac{|x^3|}{3} + \dots\right).$$

Then $$e^Y = 1 + \frac{Y}{1!} + \frac{Y^2}{2!} + \dots,$$

and this series converges for all values of Y.

Hence, on altering the order of summation, we have

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} P_n(m),$$

where $P_n(m)$ is a polynomial in $m$ of degree $n$, the term of highest degree being $m^n$. The series converges absolutely if $|x| < 1$.

Now $P_n(m)$ vanishes if $m = 0, 1, 2, \dots, n-1$. Hence

$$P_n(m) = m(m-1)(m-2)\dots(m-n+1).$$

Thus the theorem has been established.

## EXAMPLES XVIII

1. If $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$,

prove that the sequence $(u_n)$ converges to a limit between $\frac{1}{2}$ and 1.

[$u_{n+1} - u_n = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$. Hence $(u_n)$ is monotonic increasing. Also $u_n < \frac{n}{n} = 1$; thus the sequence is bounded above. Again $u_n > u_1 = \frac{1}{2}$, and

$$u_{2n} < \frac{n}{2n} + \frac{n}{3n} = \tfrac{1}{2} + \tfrac{1}{3},$$

so that the limit $\leqq \frac{5}{6} < 1$.]

2. If $u_n = \sin\left(\frac{\pi}{n+1}\right) + \sin\left(\frac{\pi}{n+2}\right) + \ldots + \sin\left(\frac{\pi}{2n}\right)$, prove that the sequence $(u_n)$ converges to a limit between $\frac{1}{2}\pi$ and $\pi$.

3. Show that

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \ldots = \tfrac{3}{2}\log 2.$$

$$\Big[s_{3n} = \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{4n} - \log 4n\right) - \tfrac{1}{2}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{2n} - \log 2n\right) - \tfrac{1}{2}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right) + \tfrac{3}{2}\log 2.$$

Now apply the property of Euler's Constant given in § 3.]

4. Prove that

(i) $1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{7} - \ldots = \tfrac{1}{2}\log 2,$
(ii) $1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{3} - \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{5} - \ldots = \tfrac{1}{2}\log \tfrac{4}{3}.$

5. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(9n^2 - 4)} = \tfrac{3}{8}(\log 3 - \tfrac{1}{2}).$$

6. Show that the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{5} - \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{12} + \tfrac{1}{7} - \ldots$$

diverges to $-\infty$.

7. If $$u_n = \frac{n}{(n+1)^2} + \frac{n+1}{(n+2)^2} + \ldots + \frac{2n-1}{(2n)^2},$$

show that the sequence $(u_n)$ converges to a limit between $\frac{1}{4}$ and 1.

8. If $$u_n = \sum_{r=1}^{\infty} \frac{a^2}{(r-a)r(r+a)},$$

where $a = p + \frac{1}{2}$ and $p$ is zero or a positive integer, show that

(i) $u_p = 2\log 2 - 2\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots + \frac{1}{2p+1}\right) + \frac{1}{2p+1},$

(ii) $u_{p+1} = u_p - \frac{1}{2p+1} - \frac{1}{2p+3}.$

9. If $n$ is a positive integer, show that

$$e^n > \frac{(n+1)^n}{n!}.$$

[Assume that the inequality holds for $n = 1, 2, 3, \ldots, n$: then

$$e^{n+1} > e\frac{(n+1)^n}{n!} = \frac{e}{\left(1+\frac{1}{n+1}\right)^{n+1}} \cdot \frac{(n+2)^{n+1}}{(n+1)!}.$$

But, from Example 5, § 1, $\left(1+\frac{1}{n+1}\right)^{n+1} < e$. Therefore

$$e^{n+1} > \frac{(n+2)^{n+1}}{(n+1)!}.$$

Now the inequality holds when $n = 1$; hence it holds for all values of $n$.]

10. Sum to $n$ terms the series $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$, and discuss the sum to infinity: *deduce* that the series $\sum \frac{1}{n^2}$ is convergent.

Ans. $S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$, $S = \frac{1}{2}$.

11. Show that

$$\lim_{n\to\infty} \sqrt[n]{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)} = 1.$$

[See Examples XVII, 88.]

12. Prove that the sum of the first $2^n$ terms of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ lies between $1 + \frac{1}{2}n$ and $n + (\frac{1}{2})^n$, and that the sum of the first $3^n$ terms lies between $1 + \frac{2}{3}n$ and $2n + (\frac{1}{3})^n$.

13. Show that the following series are divergent:

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n^2+1)}}$, (ii) $\sum_{n=1}^{\infty} \frac{1+n}{1+n^2}$, (iii) $\sum_{n=1}^{\infty} \frac{n}{n^2+6}$,

(iv) $\sum_{n=1}^{\infty} \frac{1}{n^2\{\sqrt{(n+1)} - \sqrt{n}\}^2}$.

(v) $\dfrac{1}{2^2 - 1^2} + \dfrac{1}{3^2 - 2^2} + \dfrac{1}{4^2 - 3^2} + \dfrac{1}{5^2 - 4^2} + \ldots,$

(vi) $\dfrac{5}{1.3} + \dfrac{8}{2.4} + \dfrac{11}{3.5} + \dfrac{14}{4.6} + \ldots.$

14. Show that the following series are convergent :

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^2}$,

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+n^2+1}$, (iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^2+(2n-1)^2\pi^2}$,

(v) $\dfrac{1}{1+x} + \dfrac{1}{2+x} + \dfrac{1}{2^2+x} + \dfrac{1}{2^3+x} + \ldots,$

where $x > 0$.

15. Show that tho following series are convergent :

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{3^n}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$,

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$, (iv) $\displaystyle\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$.

16. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$ is divergent.

17. Prove that the following series are absolutely convergent if $-1 < x < 1$ :

(i) $1 + \dfrac{3x}{2} + \dfrac{5x^2}{3} + \dfrac{7x^3}{4} + \ldots,$

(ii) $\dfrac{4.5}{1.2}x + \dfrac{5.6}{2.3}x^2 + \dfrac{6.7}{3.4}x^3 + \ldots,$

(iii) $1 + 2x + 3x^2 + 4x^3 + \ldots,$

(iv) $1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots,$

(v) $\frac{2}{3} + \frac{3}{5}x + \frac{5}{9}x^2 + \frac{9}{17}x^3 + \ldots + \dfrac{2^{n-1}+1}{2^n+1}x^{n-1} + \ldots,$

(vi) $\dfrac{x}{1+x^2} + \dfrac{x^2}{1+x^4} + \dfrac{x^3}{1+x^6} + \dfrac{x^4}{1+x^8} + \ldots.$

18. Test for convergency the following series :

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{(n^3+1)x^n}{n^4+1}$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(px)^{2n}}{n^p}$, where $p$ is a positive proper fraction,

(iii) $\displaystyle\sum_{n=0}^{\infty} \frac{n\,!}{(2n)\,!}x^n$, (iv) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n\,!}$,

(v) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{x^n \cdot n\,!}$, (vi) $\displaystyle\sum_{n=0}^{\infty} (n+1)(n+2)(2x)^n$,

(vii) $\displaystyle\sum_{n=1}^{\infty} e^{-nx}\cos(nx+\alpha)$,

(viii) $x - \left(\dfrac{x}{2}\right)^2 + \left(\dfrac{x}{3}\right)^3 - \left(\dfrac{x}{4}\right)^4 + \ldots$,

(ix) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx-\theta_n)}{n(n+1)}$, (x) $\displaystyle\sum_{n=0}^{\infty} \frac{(n\,!)^2}{(2n)\,!}x^n$,

(xi) $\displaystyle\sum_{n=1}^{\infty} \frac{(x^2)^{n-1}}{(x^2)^n - 1}$, (xii) $\displaystyle\sum_{n=0}^{\infty} \frac{x^p}{1+n^2x^p}$, $p < 0$.

Ans. (i) converges absolutely for $|x| \leqq 1$; (ii) for $|x| < 1/p$; (iii) and (iv) for all values of $x$; (v) for all values of $x$ except $x = 0$; (vi) for $|x| < \frac{1}{2}$; (vii) for $x > 0$; (viii) and (ix) for all values of $x$; (x) for $|x| < 4$; (xi) for $|x| < 1$; (xii) for all values of $x$ except 0.

19. If

$$f(n) = \log\left(1+\frac{x}{1}\right) + \log\left(1+\frac{x}{2}\right) + \ldots$$
$$\ldots + \log\left(1+\frac{x}{n}\right) - x\log n,$$

where $x > -1$, show, by considering the series whose $n$th term is

$$f(n) - f(n-1),$$

that $f(n)$ tends to a finite limit when $n$ tends to infinity.

20. Discuss the convergency of the series:

(i) $\sum_{n=1}^{\infty} \frac{(3x-2)^{n-1}}{n}$;

(ii) $\frac{1}{x+a} - \frac{a}{x+2a} + \frac{a^2}{x+3a} - \ldots$;

(iii) $\frac{x}{1+x^2} + \frac{x^2}{2+x^4} + \frac{x^3}{3+x^6} + \ldots$;

(iv) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{x+1}\right)^n$; (v) $\sum_{n=1}^{\infty} \frac{\sin^2 n\theta}{n^p}$.

Ans. (i) converges absolutely if $\frac{1}{3} < x < 1$ and conditionally if $x = \frac{1}{3}$; (ii) converges absolutely if $|a| < 1$ and conditionally if $a = 1$, provided that $x$ is not a negative integer; (iii) is absolutely convergent if $|x| < 1$ or $|x| > 1$ and converges conditionally if $x = -1$; (iv) converges absolutely if $x > -\frac{1}{2}$ and conditionally if $x = -\frac{1}{2}$; (v) converges absolutely if $p > 1$ and diverges if $p \leqq 1$ unless $\theta = 0$.

21. If $p > 1$, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos \frac{\theta}{n}$$

is absolutely convergent.

22. If $|x| < \frac{1}{4}$, prove that the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n \sin\left(\alpha + \frac{\beta}{n}\right)$$

is absolutely convergent.

23. For what values of $x$ is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} (\tan x)^{2n}$$

convergent?

Ans. $k\pi - \frac{1}{4}\pi \leqq x \leqq k\pi + \frac{1}{4}\pi$, where $k$ is any integer.

24. Show that the series

$$1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \ldots$$

is convergent.

25. Show that the series

$$\sum_{n=0}^{\infty}\left\{\frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}\right\}$$

converges to zero for all values of $x$.

26. Show that the series $\sum_{n=2}^{\infty}\frac{\cos n\theta}{\log n}$ is convergent if $\theta \neq 2k\pi$, where $k$ is an integer.

27. Prove that the series $\sum_{n=2}^{\infty}\frac{\sin n\theta}{n\log n}$ converges for all values of $\theta$.

28. If the sequence $(u_n)$ is monotonic decreasing and converges to zero, show that the series $\Sigma u_n \cos n\theta$ converges if $\theta$ is not a multiple (including zero) of $2\pi$, and that the series $\Sigma u_n \sin n\theta$ converges for all values of $\theta$.

29. Prove that the series

$$1+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}-\tfrac{1}{9}+\ldots$$

is divergent.

30. Prove that the series

$$\sum_{n=1}^{\infty}\cos n\theta\sin\frac{\theta}{n}$$

is convergent if $\theta \neq 2k\pi$, where $k$ is any integer except zero.

31. If, for positive integral values of $n$,

$$P_n(\cos\theta)=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}.2$$
$$\times\{a_0\cos n\theta+a_2\cos(n-2)\theta+a_4\cos(n-4)\theta+\ldots\},$$

where $a_0=1$ and

$$a_{2r}=\frac{1.3\ldots(2r-1)}{1.2\ldots r}\cdot\frac{n(n-1)\ldots(n-r+1)}{(2n-1)(2n-3)\ldots(2n-2r+1)},$$

the last coefficient, when $n$ is even, being $\tfrac{1}{2}a_n$, show that $a_{2r} < a_{2r-2}$, and deduce that, if $\theta$ is not a multiple of $\pi$,

(i) $|P_n(\cos\theta)| \leqq \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \cdot \dfrac{2}{|\sin\theta|}$,

(ii) $|P_n(\cos\theta)| < \dfrac{A}{\sqrt{n}\,.\,|\sin\theta|}$,

where A is a constant independent of $n$.

32. If the sequence $(c_n)$ is monotonic decreasing and converges to zero, show that the series $\Sigma c_n u_n$ is convergent if the series $\Sigma u_n$ is convergent or oscillates finitely.

33. Discuss the convergence of the series whose $n$th term is

(i) $\dfrac{\sqrt{(n-1)}}{\sqrt{(n^3+1)}}x^n$, (ii) $(-1)^n\dfrac{\sqrt{n}}{n+1}$, (iii) $\dfrac{1}{n}\sin\dfrac{x}{n}$.

Ans. (i) Absolutely convergent if $|x| < 1$; conditionally convergent if $x = -1$; (ii) conditionally convergent; (iii) absolutely convergent for all values of $x$.

34. Show that the series

$$1 - \frac{m}{1} + \frac{m(m-1)}{2!} - \frac{m(m-1)(m-2)}{3!} + \ldots \ldots + (-1)^n\frac{m(m-1)\ldots(m-n+1)}{n!} + \ldots$$

is absolutely convergent if $m > 0$; and that the series

$$1 + \frac{m}{1} + \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} + \ldots \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!} + \ldots$$

converges conditionally if $0 > m > -1$.

Show by induction that the sum of $n+1$ terms of the former series is

$$S_{n+1} = (-1)^n\frac{(m-1)(m-2)\ldots(m-n)}{n!},$$

and find its sum to infinity when it converges.

Ans. 0.

35. If the series $\Sigma u_n$ is absolutely convergent, and the sequence $(c_n)$ is bounded, prove that the series $\Sigma c_n u_n$ is absolutely convergent.

36. If the sequence $(c_n)$ is bounded, show that the series $\Sigma c_n x^n$ converges absolutely for $|x| < 1$.

37. If the series $\Sigma c_n$ is convergent, show that the series $\Sigma c_n x^n$ is absolutely convergent for $|x| < 1$.

38. If the series $\Sigma c_n x^n$ is convergent when $x = x_1$, show that it is absolutely convergent for $|x| < |x_1|$.

39. By applying Maclaurin's Integral Test show that the series

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1}$$

is convergent.

40. Show that the series

$$\sum_{n=0}^{\infty} c_n \cos n\theta,$$

where $c_n$ is the coefficient of $x^n$ in $F(\alpha, \beta; \gamma; x)$, is absolutely convergent if $\gamma - \alpha - \beta > 0$; and that it converges if $-1 < \gamma - \alpha - \beta \leqq 0$, provided that $\theta \neq 2m\pi$, where $m$ is any integer.

41. Show, by applying the rule for multiplication of series, that the square of the series

$$1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots$$

is equal to

$$1 + 4x + 10x^2 + \ldots + \tfrac{1}{6}(n+1)(n+2)(n+3)\,x^n + \ldots,$$

where $-1 < x < 1$.

42. Find partial fractions for

$$\frac{(a+1)(a+3)\ldots(a+2n-1)}{a(a+2)\ldots(a+2n)},$$

and hence prove that, for $-1 < x < 1$,

$$\left\{1 + \tfrac{1}{2}\frac{a}{a+2}x + \frac{1.3}{2.4}\,\frac{a}{a+4}x^2 + \ldots\right\} \times \left\{1 + \tfrac{1}{2}x + \frac{1.3}{2.4}x^2 + \ldots\right\}$$

$$= 1 + \frac{a+1}{a+2}x + \frac{(a+1)(a+3)}{(a+2)(a+4)}x^2 + \ldots.$$

Express the result in terms of the hypergeometric function.

Ans. $F(\tfrac{1}{2}, \tfrac{1}{2}a;\ \tfrac{1}{2}a + 1;\ x) \times F(\tfrac{1}{2}, 1;\ 1;\ x)$
$= F(\tfrac{1}{2}a + \tfrac{1}{2}, 1;\ \tfrac{1}{2}a + 1;\ x).$

**43.** If $|x| < 1$, show that

$$\sum_{n=1}^{\infty} \frac{x^n}{1 + x^{2n}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{1 - x^{2n-1}}.$$

[L.H.S. $= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} x^{(2m-1)n}$;

R.H.S. $= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n-1} x^{(2n-1)m}$.

Change the order of summation.]

**44.** Prove that, if $|x| < 1$,

(i) $$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{1 + x^{4n-2}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{1 - x^{4n-2}},$$

(ii) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{1 + x^{2n}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{1 + x^{2n-1}}.$$

**45.** Show that, if $|x| < 1$,

(i) $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{nx^n}{1 + x^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{(1 + x^n)^2},$$

(ii) $$\sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-1}}{1 - x^{4n-2}} = \sum_{n=1}^{\infty} \frac{x^{2n-1}(1 + x^{4n-2})}{(1 - x^{4n-2})^2}.$$

**46.** Prove that, if $|x| < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} \frac{1 + x^n}{1 - x^n} x^{n^2}.$$

[L.H.S. $= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x^{nm} = \sum_{p=1}^{\infty} c_p x^p$, where $c_p$ is the number of divisors of $p$, including 1 and $p$. Each pair of unequal divisors $r$ and $s$ such that $rs = p$ occurs twice, once when $n = r$ and $m = s$ and once when $m = r$ and $n = s$; if $r$ and $s$ are equal the pair occurs only once. Again

$$\text{R.H.S.} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{x^{n(n+m-1)} + x^{n(n+m)}\}.$$

Here each pair $r$ and $s$, if $r < s$, occurs twice, once with $r = n$ and $s = n + m - 1$ and once with $r = n$ and $s = n + m$; if $r = s$ the pair only occurs once with $r = n$ and $s = n + m - 1$ when $m = 1$.]

47. Show that, if $x > 0$ or $< -2$,

$$\frac{1}{1+x} + \frac{2}{(1+x)^2} + \frac{3}{(1+x)^3} + \ldots = \frac{1+x}{x^2}.$$

48. Use the binomial theorem to find the cube root of 1030 as far as the 7th decimal place.

Ans. 10·0990163.

49. If $|x| < 1$, find the sum of the series

$$1 + 3x + 3\tfrac{1}{2}x^2 + 4\tfrac{1}{3}x^3 + 5\tfrac{1}{4}x^4 + \ldots.$$

Ans. $(1-x)^{-2} - \log(1-x)$.

50. If $|x| < 1$, find the general term in the expansion of

$$\frac{1}{(1-x)(1-x^2)}.$$

Ans. $T_{n+1} = \frac{1}{4}\{(-1)^n + 2n + 3\}x^n$. [Find partial fractions and expand by the binomial theorem.]

51. Show that, if $x > -\frac{1}{2}$,

$$\frac{x}{\sqrt{(1+x)}} = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1.3}{2.4}\left(\frac{x}{1+x}\right)^3 + \ldots.$$

52. Find the sums of the infinite series

(i) $1 + \dfrac{1}{3} + \dfrac{1.3}{3.6} + \dfrac{1.3.5}{3.6.9} + \dfrac{1.3.5.7}{3.6.9.12} + \ldots,$

(ii) $1 + \dfrac{2}{6} + \dfrac{2.5}{6.12} + \dfrac{2.5.8}{6.12.18} + \ldots,$

(iii) $1 + \dfrac{2}{4} + \dfrac{2.5}{4.8} + \dfrac{2.5.8}{4.8.12} + \dfrac{2.5.8.11}{4.8.12.16} + \ldots.$

Ans. (i) $\sqrt{3}$; (ii) $\sqrt[3]{4}$; (iii) $2\sqrt[3]{2}$.

53. Show that the coefficient of $x^{3n}$ in the expansion in ascending powers of $x$ of

$$\frac{1-x}{(1+x+x^2)^2},$$

where $|x| < 1$, is unity.

54. Show that the coefficient of $x^{3n}$ in the expansion of $(1+x^2)^3(1-x^3)^{-2}$ in powers of $x$ is $2n$.

55. If $$u = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\,(\alpha+1)(\alpha+2)\ldots(\alpha+r)},$$

and $$v = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{s!\,(\beta+1)(\beta+2)\ldots(\beta+s)},$$

show that

$$uv = \sum_{n=0}^{\infty} \frac{(-1)^n c_n x^n}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)(\beta+1)(\beta+2)\ldots(\beta+n)},$$

where $c_n$ is the coefficient of $x^n$ in the expansion of $(1+x)^{\alpha+\beta+2n}$ in ascending powers of $x$.

56. Show that, for all values of $x$,

$$\frac{\sin x^2}{x^2 e^x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{8} \ldots .$$

57. Prove that, if $\sinh |x| < 1$,

(i) $$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} \ldots ,$$

(ii) $$\log\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)} = x + \frac{x^3}{6} + \frac{x^5}{24} \ldots .$$

58. Show that, if $e^{|x|} < 3$,

$$\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 \ldots .$$

59. If $\cosh x < 3$, show that

$$\log(1+\cos x) = \log 2 - \frac{x^2}{4} - \frac{x^4}{96} \ldots .$$

60. If $\cosh x < 2$, show that

$$\text{(i)}\ \log \cos x = -\frac{x^2}{2!} - \frac{2x^4}{4!} - \frac{16x^6}{6!} \dots,$$

$$\text{(ii)}\ \sin x \tan x = x^2 + \frac{1}{6}x^4 + \frac{31}{360}x^6 \dots,$$

$$\text{(iii)}\ \sec^2 x = 1 + x^2 + \tfrac{2}{3}x^4 + \tfrac{17}{45}x^6 \dots.$$

61. If $\cosh x < 2$, show that

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 \dots,$$

and deduce that the first term in the expansion of

$$\tan(\sin x) - \sin(\tan x) \text{ is } \tfrac{1}{30}x^7.$$

62. Prove that, if $\sinh |x| < 2|x|$,

$$\cos x \left(\frac{x}{\sin x}\right)^2 = 1 - \tfrac{1}{6}x^2 - \tfrac{7}{120}x^4 \dots.$$

63. Show that, if $|x| < \log_e 2$,

$$\tan(\tfrac{1}{4}\pi + x) = 1 + 2x + 2x^2 + \tfrac{8}{3}x^3 \dots.$$

64. If $m$ is a positive integer, and

$$f(x) = \sum_{r=1}^{m} \frac{x^r}{r},$$

show that

$$(1 - x)e^{f(x)} = 1 - \frac{x^{m+1}}{m+1} \dots.$$

65. Prove that

$$\mathop{\mathcal{L}}_{x \to 0} \frac{\log(1 + x) - x(1 + \frac{5}{6}x)^{-3/5}}{\cos x - (1 + x^2)^{-1/2}} = \tfrac{1}{36}.$$

66. If the sequence $(u_n)$ converges to a limit $l$, and if $\sum b_n$ is a divergent series of positive terms, prove that the sequence $(v_n)$, where

$$v_n = \frac{b_1u_1 + b_2u_2 + \dots + b_nu_n}{b_1 + b_2 + \dots + b_n},$$

also converges to $l$.

[*Cf.* pages 279, Example 90.]

67. Show that

$$\mathop{\mathcal{L}}_{n \to \infty} \frac{\sin\theta + \sin\frac{\theta}{2} + \dots + \sin\frac{\theta}{n}}{1 + \frac{1}{2} + \dots + \frac{1}{n}} = \theta$$

# CHAPTER XIX

# UNIFORM CONVERGENCE

## § 1. Uniform Convergence

IF the series $\Sigma u_n(x)$, in which the terms are functions of $x$, is convergent for $a < x < b$, its sum defines a function $S(x)$ in that interval. If the series contains a finite number of terms only; that is, if

$$S(x) = \sum_{r=1}^{n} u_r(x),$$

the sum possesses the following properties:

(i) if all the functions $u_r(x)$ are continuous in $(a, b)$, $S(x)$ is continuous in $(a, b)$;

(ii) the integral of the sum is the sum of the integrals; that is

$$\int_a^b S(x)dx = \sum_{r=1}^{n} \int_a^b u_r(x)dx;$$

(iii) if each of the functions $u_r(x)$ possesses a derivative at a point $x$ in $(a, b)$, so does $S(x)$, and

$$S'(x) = \sum_{r=1}^{n} u'_r(x);$$

that is, the derivative of the sum is the sum of the derivatives.

It does not necessarily follow that these properties still hold when the series contains an infinite number of terms. Conditions subject to which these theorems are valid will now be established.

*Uniformly Convergent Sequences.*—A sequence $\{u_n(x)\}$ which converges, for all values of $x$ in an interval $(a, b)$, to the limit $l(x)$, is said to be uniformly convergent in that interval if, corresponding to any assigned positive number $\epsilon$, however small, a positive integer $m$ can be found such that, if $n \geqq m$,

$$| u_n(x) - l(x) | < \epsilon$$

for all values of $x$ in the interval.

*Note.*—If the sequence is convergent in $(a, b)$, then for any particular value of $x$ in $(a, b)$ an integer $m$ can be found to correspond to $\epsilon$; but for different values of $x$ in $(a, b)$ the values of $m$ which correspond to $\epsilon$ may be different. The convergence is only *uniform* in $(a, b)$ if, corresponding to any assigned $\epsilon$, an integer $m$ can be found which applies for every value of $x$ in $(a, b)$.

*Example* 1.—Show that if, for all positive integral values of $n$,

$$u_n(x) = 1 + x + x^2 + \ldots + x^{n-1},$$

the sequence $\{u_n(x)\}$ is convergent in the open interval $(0, 1)$; but it does not converge uniformly in that interval.

If
$$0 < x < 1,$$
$$u_n(x) = \frac{1 - x^n}{1 - x}$$
and
$$l(x) = \frac{1}{1 - x}.$$
Thus
$$| u_n(x) - l(x) | = \frac{x^n}{1 - x}.$$

If $\epsilon$ is assigned, and $x$ is a fixed point in the interval, an integer $m$ can always be found such that

$$\frac{x^m}{1 - x} < \epsilon,$$
or
$$\frac{1}{x^m} > \frac{1}{\epsilon(1 - x)},$$
or
$$m > \log \left\{\frac{1}{\epsilon(1 - x)}\right\} \Big/ \log \left(\frac{1}{x}\right).$$

As $x$ tends to 1, the expression on the right tends to infinity. Thus no value of $m$ can be found such that $|u_m(x) - l(x)| < \epsilon$ for *all* values of $x$ in the open interval (0, ·1). Hence the sequence, though convergent in that interval, is not *uniformly* convergent in the interval.

*Example* 2.—If, for all positive integral values of $n$,

$$u_n(x) = \frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \ldots$$
$$\ldots + \frac{1}{(x+n-1)(x+n)},$$

show that the sequence $\{u_n(x)\}$ is uniformly convergent for $x > 0$.

$$u_n(x) = \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1}$$
$$- \frac{1}{x+1} - \frac{1}{x+2} - \ldots - \frac{1}{x+n}$$
$$= \frac{1}{x} - \frac{1}{x+n}.$$

Hence $$l(x) = \frac{1}{x},$$

and $$|u_n(x) - l(x)| = \frac{1}{x+n} < \frac{1}{n}.$$

Now choose $m$ so large that $m > 1/\epsilon$; then, if $n \geqq m$,

$$|u_n(x) - l(x)| < \epsilon, \quad x > 0.$$

*Example* 3.—If the sequence $(u_n)$ is convergent in the closed interval $(a, b)$, and is uniformly convergent in the open interval $(a, b)$, prove that it is uniformly convergent in the closed interval $(a, b)$.

Let $m'$ correspond to $\epsilon$ for $a < x < b$, and let $m_1$ and $m_2$ correspond to $\epsilon$ for $x = a$ and $x = b$ respectively. Then the greatest of $m'$, $m_1$, $m_2$ corresponds to $\epsilon$ for $a \leqq x \leqq b$.

*Uniformly Convergent Series.*—The series

$$\Sigma u_n(x) \equiv u_1(x) + u_2(x) + u_3(x) + \ldots$$

is said to be uniformly convergent in an interval $(a, b)$ if the sequence $\{S_n(x)\}$, where $S_n(x)$ is the sum of the first $n$ terms of the series, is uniformly convergent in that interval.

That is to say, if $S(x)$ is the sum of the series, and

$$R_n(x) = S(x) - S_n(x),$$

the series is uniformly convergent in $(a, b)$ if, corresponding to any assigned positive number $\epsilon$, however small, a positive integer $m$ can be found such that, if $n \geqq m$,

$$| R_n(x) | < \epsilon,$$

for every value of $x$ in $(a, b)$.

*Example* 4.—Show that the series $1 + x + x^2 + x^3 + \ldots$ is convergent, but not uniformly convergent, in the open interval $(0, 1)$.

*Example* 5.—Show that the series

$$\frac{1}{x^2(x^2+1)} + \frac{1}{(x^2+1)(x^2+2)} + \frac{1}{(x^2+2)(x^2+3)} + \ldots$$

is uniformly convergent for all values of $x$ except $x = 0$.

*Example* 6.—Show that the series

$$\frac{1}{x(x+1)} + \frac{2}{(x+1)(x+3)} + \frac{3}{(x+3)(x+6)} + \frac{4}{(x+6)(x+10)} + \ldots$$

is uniformly convergent for $x > 0$.

*Note.*—In what follows it will be assumed, unless it is expressly stated to be otherwise, that the interval of uniform convergence is *closed*, that is, $a \leqq x \leqq b$. This is a convenient, though not a necessary, assumption.

THEOREM I.—If all the terms of the series

$$S(x) = \sum_{n=1}^{\infty} u_n(x),$$

which is uniformly convergent in a closed interval $(a, b)$, are continuous in that interval, the sum $S(x)$ is continuous in the interval.

For, let $x_1$ be any point of the interval and $x$ a neighbouring point (of the interval). Then, corresponding to any assigned $\epsilon$, a positive integer $m$ can be found such that, if $n \geqq m$,

$$| S(x_1) - S_n(x_1) | < \tfrac{1}{3}\epsilon, \quad | S(x) - S_n(x) | < \tfrac{1}{3}\epsilon.$$

Now fix on one such value of $n$. Then, since $S_n(x)$ is the sum of $n$ continuous functions, it is itself a continuous function. Thus, corresponding to the assigned $\epsilon$, a positive number $\eta$ can be found such that, if $| x - x_1 | < \eta$,

$$| S_n(x) - S_n(x_1) | < \tfrac{1}{3}\epsilon.$$

Hence, if $| x - x_1 | < \eta$,

$$\begin{aligned} &| S(x) - S(x_1) | \\ &= | \{S(x) - S_n(x)\} - \{S(x_1) - S_n(x_1)\} + \{S_n(x) - S_n(x_1)\} | \\ &\leqq | S(x) - S_n(x) | + | S(x_1) - S_n(x_1) | + | S_n(x) - S_n(x_1) | \\ &< \epsilon. \end{aligned}$$

Thus $S(x)$ is continuous at $x_1$, any point of $(a, b)$. Therefore $S(x)$ is continuous in $(a, b)$.

*Example* 7.—Show that the series

$$x^2 + \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} + \ldots$$

is convergent for all real values of $x$, but that it is not uniformly convergent in any interval containing the point $x = 0$.

If $x \neq 0$ the series is a convergent geometrical progression with sum $1 + x^2$. If $x = 0$ each term has the value zero and therefore the sum is zero. Thus the sum $S(x)$ is discontinuous at $x = 0$; and, consequently, the series cannot be uniformly convergent in an interval containing $x = 0$.

*Example* 8.—Show that the series

$$(1 - x) + (x - x^2) + (x^2 - x^3) + \ldots$$

converges at all points of the closed interval $(0, 1)$, but that it does not converge uniformly in that interval.

*Example* 9.—Show that the series

$$x + xe^{-x} + xe^{-2x} + xe^{-3x} + \ldots$$

is convergent but not uniformly convergent for $0 \leqq x$.

*Example* 10.—Show that the series $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ is convergent but not uniformly convergent in any interval containing $x = 0$. [*Cf*. Ch. XVIII, § 3, Example 6.]

THEOREM II.—If $\alpha$ and $\beta$ are any points in a closed interval $(a, b)$ in which the series *

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

converges uniformly, the series

$$\sum_{r=1}^{\infty} \int_{\alpha}^{\beta} u_r(x)dx$$

converges to the sum $\int_{\alpha}^{\beta} S(x)dx$.

In the first place, assume that $\alpha < \beta$. Then since, at all points of the interval,

$$| S(x) - S_n(x) | < \epsilon, \quad n \geqq m,$$

for any such value of $n$

$$\left| \int_{\alpha}^{\beta} S(x)dx - \sum_{r=1}^{n}\int_{\alpha}^{\beta} u_r(x)dx \right| = \left| \int_{\alpha}^{\beta} \{S(x) - S_n(x)\}dx \right|$$

$$\leqq \int_{\alpha}^{\beta} | S(x) - S_n(x) | \, dx$$

$$< \int_{\alpha}^{\beta} \epsilon \, dx = \epsilon(\beta - \alpha) \leqq \epsilon(b - a).$$

But $\epsilon(b - a)$ can be chosen as small as we please. Hence the series of integrals converges to the sum $\int_{\alpha}^{\beta} S(x)dx$.

* It is assumed that each function $u_n(x)$ is continuous in $(a, b)$.

If $\alpha > \beta$

$$\left| \int_\alpha^\beta S(x)dx - \sum_{r=1}^{n} \int_\alpha^\beta u_r(x)dx \right|$$
$$= \left| \int_\beta^\alpha S(x)dx - \sum_{r=1}^{n} \int_\beta^\alpha u_r(x)dx \right| < \epsilon(b-a),$$

so that the result holds in this case also.

Moreover, the series of functions of $x$,

$$\sum_{n=1}^{\infty} \int_\alpha^x u_n(x)dx$$

converges uniformly in $(a, b)$. For, if $n \geqq m$,

$$\left| \int_\alpha^x S(x)dx - \sum_{r=1}^{n} \int_\alpha^x u_r(x)dx \right| < \epsilon(b-a)$$

for all values of $x$ in the interval.

THEOREM III.—If the series*

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

is convergent in a closed interval $(a, b)$ and if the series $\Sigma u'_n(x)$ is uniformly convergent in that interval, $S(x)$ possesses a derivative $S'(x)$, and

$$S'(x) = \Sigma u'_n(x).$$

For, by Theorem II, if $\alpha$ and $x$ are points of $(a, b)$, and if

$$F(x) = \Sigma u'_n(x),$$

$$\begin{aligned}\int_\alpha^x F(x)dx &= u_1(x) + u_2(x) + u_3(x) + \cdots \\ &\quad - u_1(\alpha) - u_2(\alpha) - u_3(\alpha) - \cdots \\ &= S(x) - S(\alpha);\end{aligned}$$

or

$$S(x) = \int_\alpha^x F(x)dx + S(\alpha).$$

* It is assumed that each function $u_n(x)$ is continuous in $(a, b)$.

But the R.H.S. has a derivative $F(x)$; hence

$$S'(x) = F(x) = \Sigma u'_n(x).$$

*The Partial Remainder.**—In the definition of uniform convergence the condition

$$| R_n(x) | < \epsilon, \quad n \geqq m,$$

may be replaced by the condition

$$| {}_pR_n(x) | < \epsilon, \quad n \geqq m, \quad p = 1, 2, 3, \ldots$$

The equivalence of the two conditions may be established as follows.

In the first place, let it be given that

$$| R_n(x) | < \tfrac{1}{2}\epsilon, \quad n \geqq m.$$

Then, if $p = 1, 2, 3, \ldots,$

$$\begin{aligned} | {}_pR_n(x) | &= | S_{n+p}(x) - S_n(x) | \\ &= | \{S(x) - S_n(x)\} - \{S(x) - S_{n+p}(x)\} | \\ &= | R_n(x) - R_{n+p}(x) | \\ &\leqq | R_n(x) | + | R_{n+p}(x) | < \epsilon. \end{aligned}$$

In the next place, let it be given that

$$| {}_pR_n(x) | < \epsilon_1, \quad n \geqq m, \quad p = 1, 2, 3, \ldots,$$

where $\epsilon_1 < \epsilon$.

Then
$$R_n(x) = \lim_{p \to \infty} {}_pR_n(x),$$

and therefore

$$| R_n(x) | \leqq \epsilon_1 < \epsilon.$$

*Example* 11.—Show that the series

$$\text{(i)} \sum_{n=1}^{\infty} \frac{\cos n\theta}{n}, \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$$

converge uniformly in the interval $\alpha \leqq \theta \leqq 2\pi - \alpha$, where $\alpha$ is a small positive number.

* *Cf.* Ch. XVIII, § 2.

For (i), as in Example 1 of Chapter XVIII, § 5,

$$|\,{}_pR_n(\theta)\,| \leq \frac{1}{(n+1)\,|\sin\frac{1}{2}\theta\,|} \leqq \frac{1}{(n+1)\sin\frac{1}{2}\alpha}$$

if $\alpha \leqq \theta \leqq 2\pi - \alpha$. Now choose $m$ so large that

$$\frac{1}{(m+1)\sin\frac{1}{2}\alpha} < \epsilon.$$

Then, if $n \geq m$,

$$|\,{}_pR_n\,(\theta)\,| < \epsilon\,;$$

and, consequently, the series is uniformly convergent in the interval $(\alpha, 2\pi - \alpha)$.

The uniform convergence of (ii) can be established in the same manner.

*Example* 12.—Prove that the series

$$\frac{1-\cos 2\theta}{1} + \frac{\cos 2\theta - \cos 4\theta}{3} + \frac{\cos 4\theta - \cos 6\theta}{5} + \ldots$$

converges uniformly in any interval. Deduce that the series

$$\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \ldots$$

converges uniformly in the interval $\alpha \leqq \theta \leqq \pi - \alpha$, where $0 < \alpha < \frac{1}{2}\pi$.

*Example* 13.—If each element of the sequence $\{u_n(x)\}$ is continuous in the closed interval $(a, b)$, and if the sequence converges uniformly in that interval to the limit $s(x)$, prove that the function $s(x)$ is continuous in the interval.

*Example* 14.—If, in Example 13, $\alpha$ and $x$ are any points of $(a, b)$, and if

$$v_n(x) = \int_\alpha^x u_n(x)dx, \quad n = 1, 2, 3, \ldots,$$

show that the sequence $\{v_n(x)\}$ converges uniformly in $(a, b)$ to the limit

$$\int_\alpha^x s(x)dx.$$

*Example* 15.—If the sequence $\{u_n(x)\}$ is convergent and the sequence $\{u'_n(x)\}$ uniformly convergent in $(a, b)$, prove that $s(x)$ possesses a derivative $s'(x)$ and that $s'(x)$ is the limit of $\{u'_n(x)\}$ in $(a, b)$.

## § 2. Weierstrass's M-Test

If, corresponding to a given series $\Sigma u_n(x)$, a convergent series of positive numbers $\Sigma M_n$ can be found such that, for all values of $n$ and for $a \leqq x \leqq b$, $| u_n(x) | \leqq M_n$, the series $\Sigma u_n(x)$ is absolutely and uniformly convergent in the closed interval $(a, b)$.

For, since $\Sigma M_n$ is convergent, a positive integer $m$ can be found such that, if $n \geqq m$,

$$M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \epsilon, \quad p = 1, 2, 3, \ldots$$

Hence, if $a \leqq x \leqq b$,

$$|{}_pR_n(x)| \leqq |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots + |u_{n+p}(x)|$$
$$\leqq M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \epsilon.$$

*Example* 1.—Show that the series

$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \ldots$$

converges absolutely and uniformly in the closed interval $(-1, 1)$. [Take $M_n = n^{-2}$.]

*Example* 2.—If $|r| < 1$, show that

$$\int_0^\pi \frac{\theta \sin \theta \, d\theta}{1 - 2r \cos \theta + r^2} = \frac{\pi}{r} \log (1 + r).$$

[See Ch. XVIII, § 4, Example 3 (ii). Take $M_n = |r|^n$.]

*Example* 3.—Prove that the sum of the series

$$\frac{1}{\theta} + \sum_{r=1}^{\infty} \frac{2\theta}{\theta^2 - r^2\pi^2}$$

is a continuous function of $\theta$ except at the points $\theta = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$

Let $m$ be any positive integer, and assume that

$$-m\pi \leqq \theta \leqq m\pi.$$

Then, if $r > m$,

$$|\theta \pm r\pi| \geqq r\pi - |\theta| \geqq (r - m)\pi.$$

Hence

$$\left|\frac{\theta}{\theta^2 - r^2\pi^2}\right| \leqq \frac{m\pi}{(r-m)^2\pi^2} = \frac{m}{(r-m)^2\pi}, \quad r > m.$$

But the series

$$\sum_{r=m+1}^{\infty} \frac{m}{(r-m)^2\pi} = \frac{m}{\pi}\sum_{p=1}^{\infty}\frac{1}{p^2},$$

where $p = r - m$, is a convergent series of positive terms. Hence the series

$$\sum_{r=m+1}^{\infty} \frac{\theta}{\theta^2 - r^2\pi^2}$$

converges absolutely and uniformly for $|\theta| \leqq m\pi$. It therefore represents a continuous function of $\theta$ in the closed interval $(-m\pi, m\pi)$.

Thus the function

$$\frac{1}{\theta} + \sum_{r=1}^{\infty} \frac{2\theta}{\theta^2 - r^2\pi^2}$$

is a continuous function of $\theta$ in that interval, except when $\theta = n\pi$, $n = 0, \pm 1, \pm 2, \ldots, \pm m$. But, for any value of $\theta$, $m$ can be chosen so large that $|\theta| \leqq m\pi$. Hence the result holds for all values of $\theta$.

*Example* 4.—Prove that the sum of the series

$$\frac{1}{\theta} + \sum_{n=1}^{\infty}\left(\frac{1}{\theta - n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\theta + n\pi} - \frac{1}{n\pi}\right)$$

is a continuous function of $\theta$ except at the points $\theta = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$

*Example* 5.—If $u_n(x) = x^n/n^2$ for $-1 < x < 1$, and if $u_n(x) = 1$ for $x = \pm 1$, show that the series $\Sigma u_n(x)$ converges uniformly in the open interval $(a, b)$, but not in the closed interval $(a, b)$.

## § 3. Power Series

Let R be the radius of convergence of the power series

$$\Sigma a_n x^n \equiv a_0 + a_1 x + a_2 x^2 + \ldots.$$

Then (Ch. XVIII, § 4), if $0 < R_1 < R$, the series

$$|a_0| + |a_1| \,.\, R_1 + |a_2| \,.\, R_1^2 + |a_3| \,.\, R_1^3 + \ldots$$

is a convergent series of positive terms. Hence, by Weierstrass's M-Test, $\Sigma a_n x^n$ is absolutely and uniformly convergent in the closed interval $(-R_1, R_1)$.

*Note.*—If $x_1$ is any value of $x$ in the open interval $(-R, R)$; that is, if $|x_1| < R$, $R_1$ can be chosen so that

$$|x_1| < R_1 < R;$$

thus $x_1$ can always be included in an interval of uniform convergence.

The following three corollaries follow from Theorems I, II and III of § 1.

Corollary I.—If $|x| < R$ the function $S(x) = \Sigma a_n x^n$ is continuous at $x$; that is, $S(x)$ is continuous in the open interval $(-R, R)$.

Corollary II.—If $|x| < R$,

$$\int_0^x S(x)\,dx = a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \ldots;$$

that is, the integral of the sum is the sum of the integrals of the terms.

Corollary III.—If $|x| < R$, $S(x)$ possesses a derivative $S'(x)$, and

$$S'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots;$$

that is, the derivative of the sum is the sum of the derivatives of the terms.

For the last series has the same radius of convergence as $\Sigma a_n x^n$, since

$$\mathop{\mathcal{L}}_{n\to\infty}\left|\frac{na_n}{(n+1)a_{n+1}}\right| = \mathop{\mathcal{L}}_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R.$$

Hence $x$ can always be included in an interval of uniform convergence $(-R_1, R_1)$ of the series of derivatives.

*Example* I.—Show that the series $1 + x + x^2 + \ldots$ converges uniformly in the closed interval $(-R, R)$, where $0 < R < 1$. Show also that the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots$$

converges uniformly in the same interval, and that its sum is $(1-x)^{-2}$.

*Example* 2.—Show that the series for $\exp(x)$ converges uniformly in the closed interval $(-R, R)$, where R is any positive number. Deduce that, for all values of $x$,

$$D \exp(x) = \exp(x).$$

*Example* 3.—If $-1 < x < 1$, integrate the expansions

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots,$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots.$$

over the range $(0, x)$, and show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad |x| < 1,$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots,$$

where $|x| < 1, \quad |\tan^{-1} x| < \frac{\pi}{4}$.

*Example* 4.—If $-1 < x < 1$, show that

$$\frac{d}{dx}F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1; \gamma+1; x).$$

*Example* 5.—If $|r| < 1$, prove that

$$\text{(i)}\ \log(1 - 2r\cos\theta + r^2) = -2\sum_{n=1}^{\infty} \frac{r^n}{n}\cos n\theta,$$

$$\text{(ii)}\ \tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) = \sum_{n=1}^{\infty} \frac{r^n}{n}\sin n\theta,$$

where the inverse tangent has its principal value (Ch. IV, § 2).

[For (i) integrate equation (iv) of Example 3, Ch. XVIII, § 4, with regard to $r$. For (ii) divide equation (ii) of the same example by $r$ before integrating.]

*Example* 6.—If $0 < \theta < \pi$, show that

$$\log(\operatorname{cosec}\theta) = \sum_{n=1}^{\infty} (\cos\theta)^n \frac{\cos n\theta}{n}.$$

*Example* 7.—If $-\frac{1}{2}\pi < \sin^{-1} x < \frac{1}{2}\pi$, show that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \dots$$

[Integrate the equation

$$\frac{1}{\sqrt{(1 - x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots,$$

in which $|x| < 1$, from 0 to $x$. See also § 4, Example 4.]

*Example* 8.—If $-\frac{1}{2}\pi < \sin^{-1} x < \frac{1}{2}\pi$, show that

$$\tfrac{1}{2}(\sin^{-1} x)^2 = \frac{x^2}{2\,!} + 2^2\frac{x^4}{4\,!} + 2^2 \cdot 4^2\frac{x^6}{6\,!} + \dots$$

[The series for $\sin^{-1} x$ and $1/\sqrt{(1 - x^2)}$ converge absolutely if $|x| < 1$. Thus, if $f(x) = \sin^{-1} x/\sqrt{(1 - x^2)}$, there is an expansion

$$f(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1},$$

convergent for $|x| < 1$. On differentiating this equation $2n + 1$ times and putting $x = 0$ we find that

$$f^{(2n+1)}(0) = a_{2n+1} \cdot (2n + 1)\,!$$

Now
$$\sqrt{(1 - x^2)}\, f(x) = \sin^{-1} x,$$

and this, on differentiation, gives

$$\sqrt{(1 - x^2)} f'(x) - \frac{x}{\sqrt{(1 - x^2)}} f(x) = \frac{1}{\sqrt{(1 - x^2)}},$$

or
$$(1 - x^2) f'(x) - x f(x) = 1.$$

Next, differentiate this equation $n$ times, applying Leibnitz's Theorem.

Thus $(1 - x^2) f^{(n+1)}(x) - (2n + 1)x f^{(n)}(x) - n^2 f^{(n-1)}(x) = 0,$

and therefore

$$f^{(n+1)}(0) = n^2 f^{(n-1)}(0).$$

But $f(0) = 0$, $f'(0) = 1$; hence, if $n$ is even, $f^{(n)}(0) = 0$, while

$$f^{(2n+1)}(0) = (2n)^2 (2n - 2)^2 \dots 4^2 \cdot 2^2.$$

It follows that

$$\sin^{-1} x/\sqrt{(1 - x^2)} = \frac{x}{1\,!} + 2^2\frac{x^3}{3\,!} + 2^2 \cdot 4^2\frac{x^5}{5\,!} + \dots,$$

and from this the required formula can be deduced by integrating from 0 to $x$. See also § 4, Example 4.]

## § 4. Abel's Theorem

If the series

$$a_0 + a_1 x + a_2 x^2 + \ldots$$

converges for $-1 < x \leqq 1$, it is uniformly convergent in the closed interval (0, 1).

For, since the series $\Sigma a_n$ is convergent, a positive integer $m$ can be found such that, if $n \geqq m$,

$$-\epsilon < a_{n+1} + a_{n+2} + \ldots + a_{n+p} < \epsilon,$$

where $p = 1, 2, 3, \ldots$.

Now, if $0 \leqq x \leqq 1$, $x^n$ does not increase as $n$ increases; hence, by Abel's Inequality,

$$-\epsilon x^{n+1} \leqq a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \ldots + a_{n+p}x^{n+p} \leqq \epsilon x^{n+1};$$

or, since $0 \leqq x^{n+1} \leqq 1$,

$$-\epsilon \leqq a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \ldots + a_{n+p}x^{n+p} \leqq \epsilon,$$

where $n \geqq m$, $p = 1, 2, 3, \ldots$, and $0 \leqq x \leqq 1$.

Thus the series converges uniformly in the closed interval (0, 1).

If $S(x)$ is the sum of the series, it follows that $S(x)$ is continuous for $-1 < x \leqq 1$; or, that $S(x)$ tends to $S(1)$ as $x$ tends to 1 from the left.

Corollary.—If the series converges for $-R < x \leqq R$, it converges uniformly in the closed interval (0, R). This extension of Abel's Theorem can be deduced from the theorem as stated above by means of the substitution $x = \xi R$; the resulting series in $\xi$ converges for $-1 < \xi \leqq 1$. Similarly, if the series converges for $-R \leqq x < R$, it converges uniformly in the closed interval $(-R, 0)$. This is deduced by means of the substitution $x = -\xi R$.

*Example* 1.—Show that the expansion

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

is valid for $-1 < x \leqq 1$.

In Example 3, § 3, the expansion was established for $-1 < x < 1$. But the series converges when $x = 1$. Hence, by Abel's Theorem, the sum is continuous at $x = 1$, and is therefore equal to $\lim\limits_{x \to 1} \log(1 + x)$. Thus

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots .$$

*Example* 2.—Show that the expansion

$$\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

is valid for $-1 \leqq x \leqq 1$, $-\frac{1}{4}\pi \leqq \tan^{-1} x \leqq \frac{1}{4}\pi$. Deduce that

$$\frac{\pi}{4} = \tfrac{1}{1} - \tfrac{1}{3} + \tfrac{1}{5} - \dots .$$

*Example* 3.—Prove that, if $\theta \neq 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$,

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\log |\, 2 \sin \tfrac{1}{2}\theta \,|,$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \tan^{-1} (\cot \tfrac{1}{2}\theta),$$

where the inverse tangent has its principal value.

From (ii) deduce that

$$\text{(iii)} \quad \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \tfrac{1}{2}(\pi - \theta), \quad 0 < \theta < 2\pi,$$

$$\text{(iv)} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin n\theta}{n} = \tfrac{1}{2}\theta, \quad -\pi < \theta < \pi.$$

[The series given in § 3, Example 5, are convergent for $r = 1$, provided that $\theta \neq 2n\pi$. Let $r \to 1$ and apply Abel's Theorem.]

*Example* 4.—Show that, in Examples 7, 8 of § 3, the expansions are valid for $-\frac{1}{2}\pi \leqq \sin^{-1} x \leqq \frac{1}{2}\pi$, and deduce that

$$\text{(i)} \quad \frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \,.\, 3}{2 \,.\, 4} \cdot \frac{1}{5} + \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6} \cdot \frac{1}{7} + \dots,$$

$$\text{(ii)} \quad \frac{\pi^2}{8} = \frac{1}{2\,!} + \frac{2^2}{4\,!} + \frac{2^2 \,.\, 4^2}{6\,!} + \frac{2^2 \,.\, 4^2 \,.\, 6^2}{8\,!} + \dots .$$

## § 5. Identities

The two following theorems are required in the proofs of various trigonometrical expansions.

THEOREM I.—If the series

$$\phi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

is convergent for $-R < x < R$, and if the function $\phi(x)$ vanishes for every value of $x$ in the interval, the series vanishes identically; that is to say, all the coefficients $a_0, a_1, a_2, \dots$ have the value zero.

From the given expansion $\phi(0) = a_0$; but $\phi(0)$ has the value zero; hence $a_0 = 0$.

Again, since $\phi(x) = 0$ for $-R < x < R$, it follows that $\phi'(x) = 0$ for $-R < x < R$. But, in the same interval,

$$\phi'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots.$$

Therefore $a_1 = 0$.

Similarly, by differentiating repeatedly and putting $x = 0$ it can be shown that all the coefficients vanish.

THEOREM II.—If the series

$$\phi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots,$$
$$\psi(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$$

are both convergent for $-R < x < R$, and if the functions $\phi(x)$ and $\psi(x)$ are equal for all values of $x$ in the interval, the series are identically equal; that is

$$a_n = b_n, \quad n = 0, 1, 2, 3, \dots.$$

For, since the function $\{\phi(x) - \psi(x)\}$ vanishes for all values of $x$ in the interval, the series

$$\sum_{n=0}^{\infty} (a_n - b_n)x^n$$

vanishes identically.

*Example* 1.—If $n$ is a positive integer, show that

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \dots$$
$$+ (-1)^p \frac{n(n-p-1)(n-p-2)\dots(n-2p+1)(2\cos\theta)^{n-2p}}{p!} + \dots.$$

If $|r| < 1$ (Example 3 (iii), § 4, Ch. XVIII),

$$\frac{1-r^2}{1-2r\cos\theta+r^2} = 1 + 2r\cos\theta + 2r^2\cos 2\theta + 2r^3\cos 3\theta + \dots.$$

Now, if $|r|$ is so small that $|2r\cos\theta - r^2| < 1$, the L.H.S. can be expanded in the form

$$\frac{1-r^2}{1-2r\cos\theta+r^2} = (1-r^2)\sum_{n=0}^{\infty}(2r\cos\theta - r^2)^n$$

and (Ch. XVIII, § 7) rearranged in powers of $r$. The coefficients of $r^n$ in the two series can then, by Theorem II, be equated, so giving

$$2\cos n\theta = (2\cos\theta)^n - \frac{n-1}{1!}(2\cos\theta)^{n-2} + \frac{(n-2)(n-3)}{2!}(2\cos\theta)^{n-4} - \frac{(n-3)(n-4)(n-5)}{3!}(2\cos\theta)^{n-6} + \dots$$
$$\dots - (2\cos\theta)^{n-2} + \frac{n-3}{1!}(2\cos\theta)^{n-4} - \frac{(n-4)(n-5)}{2!}(2\cos\theta)^{n-6} + \dots$$
$$= (2\cos\theta)^n - \frac{n}{1!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \frac{n(n-4)(n-5)}{3!}(2\cos\theta)^{n-6} + \dots.$$

*Example* 2.—If $n$ is a positive integer, show that

(i) $$\frac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - \frac{n-2}{1!}(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{2!}(2\cos\theta)^{n-5} - \frac{(n-4)(n-5)(n-6)}{3!}(2\cos\theta)^{n-7} + \dots,$$

(ii) $$\frac{\cos(2n+1)\theta}{\cos\theta} = 1 - \frac{n(n+1)}{2!}(2\sin\theta)^2 + \frac{(n-1)n(n+1)(n+2)}{4!}(2\sin\theta)^4 - \dots$$

[Employ (ii) and (v) of Example 3, Ch. XVIII, § 4.]

## § 6. Linear Differential Equations of the Second Order

Various expansions for trigonometrical functions can be established by transforming the differential equations which they satisfy and solving the transformed equations in series.

Two solutions of a differential equation are said to be *linearly independent* if their quotient is not a constant.

The following theorem holds for all linear equations of the second order.

THEOREM.—If $y_1$ and $y_2$ are linearly independent solutions of the equation

$$p\frac{d^2y}{dx^2} + q\frac{dy}{dx} + ry = 0,$$

where $p$, $q$, $r$ are functions of $x$, any other solution $y$ can be expressed in the form

$$y = Ay_1 + By_2,$$

where A and B are constants, not both zero.

It is sufficient for our purpose to prove this theorem for the equation

$$y'' + qy' + ry = 0, \quad . \quad . \quad . \quad (1)$$

in which $q$ and $r$ are continuous near $x = 0$, and to confine ourselves to the case in which the three solutions can be expressed as series of positive integral powers of $x$, convergent for $|x| < k$, where $k$ is a constant.

The theorem then follows from the two following lemmas:

LEMMA I.—If a solution $y(x)$ of (1) and its first derivative $y'(x)$ vanish when $x = 0$, it vanishes identically.

For, if $y(0)=0$ and $y'(0)=0$, the equation gives $y''(0)=0$. Similarly, on differentiating the equation repeatedly, we find that $y'''(0)=0$, $y''''(0)=0$, and so on. Hence, if

$$y=\sum_{n=0}^{\infty} a_n x^n,$$

the coefficients $a_0, a_1, a_2, \ldots$ are all zero, and the solution vanishes identically.

Lemma II.—If $y_1, y_2, y_3$ are solutions of the equation (1) which do not vanish identically, a relation of the form

$$c_1y_1+c_2y_2+c_3y_3=0$$

exists, where $c_1, c_2, c_3$ are constants, not more than one being zero.

For $c_1, c_2, c_3$ can always be chosen to satisfy the two equations *

$$c_1y_1(0)+c_2y_2(0)+c_3y_3(0)=0,$$
$$c_1y_1'(0)+c_2y_2'(0)+c_3y_3'(0)=0,$$

and therefore, by Lemma I, the solution $c_1y_1+c_2y_2+c_3y_3$ vanishes identically.

The theorem can now be established. From Lemma II,

$$cy=c_1y_1+c_2y_2,$$

not more than one of the constants $c, c_1, c_2$ being zero. But $c$ cannot be zero, as $y_1$ and $y_2$ are linearly independent.

Hence

$$y=\mathrm{A}y_1+\mathrm{B}y_2,$$

where

$$\mathrm{A}=c_1/c, \quad \mathrm{B}=c_2/c.$$

* If $y_1(0)$, $y_2(0)$, $y_3(0)$ are all zero, or if $y'_1(0)$, $y'_2(0)$, $y'_3(0)$ are all zero, any values of $c_1, c_2, c_3$ which satisfy the remaining equation will do. If the ratios $y'_1(0)/y_1(0)$, $y'_2(0)/y_2(0)$, $y'_3(0)/y_3(0)$ are all equal, any values $c_1, c_2, c_3$ which satisfy one equation will satisfy the other. In all other cases, $c_1, c_2, c_3$ can be found from the equations.

As an illustration of the method, consider the equation

$$\frac{d^2y}{dx^2} + n^2y = 0, \quad . \quad . \quad . \quad (2)$$

of which $\cos nx$ and $\sin nx$ are independent solutions. Put $u = \sin^2 x$, and transform the equation so that $u$ becomes the independent variable. Then

$$\frac{dy}{dx} = \frac{dy}{du} 2 \sin x \cos x,$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d^2y}{du^2}(2 \sin x \cos x)^2 + \frac{dy}{du} 2(\cos^2 x - \sin^2 x) \\ &= \frac{d^2y}{du^2} 4u(1-u) + \frac{dy}{du} 2(1-2u),\end{aligned}$$

so that equation (2) becomes

$$u(1-u)\frac{d^2y}{du^2} + (\tfrac{1}{2} - u)\frac{dy}{du} + \tfrac{1}{4}n^2y = 0. \quad . \quad (3)$$

This equation will now be solved in series of powers of $u$. Assume that there is a convergent series

$$y = u^\rho \sum_{\nu=0}^{\infty} c_\nu u^\nu \quad . \quad . \quad . \quad (4)$$

which satisfies the equation. As the series is convergent, it may be differentiated term by term, and consequently

$$\frac{dy}{du} = u^{\rho-1} \sum_{\nu=0}^{\infty} c_\nu (\rho + \nu) u^\nu,$$

and

$$\frac{d^2y}{du^2} = u^{\rho-2} \sum_{\nu=0}^{\infty} c_\nu (\rho + \nu)(\rho + \nu - 1) u^\nu.$$

When these series and (4) are substituted in (3), that equation becomes

$$u^{\rho-1}\sum_{\nu=0}^{\infty} c_\nu(\rho+\nu)(\rho+\nu-\tfrac{1}{2})u^\nu - u^\rho \sum_{\nu=0}^{\infty} c_\nu\{(\rho+\nu)^2-\tfrac{1}{4}n^2\}u^\nu = 0.$$

In order that this equation may be satisfied, the coefficients of all the powers of $u$ in it must vanish; thus the equations

$$c_0\rho(\rho-\tfrac{1}{2})=0, \quad . \quad . \quad . \quad (5)$$

and

$$c_{\nu+1}(\rho+\nu+1)(\rho+\nu+\tfrac{1}{2}) = c_\nu\{(\rho+\nu)^2-\tfrac{1}{4}n^2\}, \quad (6)$$

where $\nu = 0, 1, 2, 3, \ldots$, must all be satisfied. The equations (6) will all be satisfied if the coefficients $c_\nu$ are connected by the equations

$$c_{\nu+1} = c_\nu\frac{(\rho+\nu)^2-\tfrac{1}{4}n^2}{(\rho+\nu+1)(\rho+\nu+\tfrac{1}{2})}, \quad . \quad . \quad (7)$$

where $\nu = 0, 1, 2, 3, \ldots$.

*The Indicial Equation.*—As $c_0$, being the coefficient of the first term in (4), does not vanish, equation (5) is satisfied only if $\rho = 0$ or $\rho = \tfrac{1}{2}$. These values of $\rho$ give the only two possible indices of the lowest power of $u$ in (4). For this reason equation (5) is called the Indicial Equation.

There are then two cases to be considered, corresponding to the two roots of the indicial equation.

CASE I.—Let $\rho = 0$; then

$$c_{\nu+1} = c_\nu\frac{\nu^2-\tfrac{1}{4}n^2}{(\nu+1)(\nu+\tfrac{1}{2})} = c_\nu\frac{(\tfrac{1}{2}n+\nu)(-\tfrac{1}{2}n+\nu)}{(\tfrac{1}{2}+\nu)(\nu+1)},$$

so that

$$y = c_0\left\{1 + \frac{(\tfrac{1}{2}n)(-\tfrac{1}{2}n)}{\tfrac{1}{2}\,.\,1}u + \frac{(\tfrac{1}{2}n)(\tfrac{1}{2}n+1)(-\tfrac{1}{2}n)(-\tfrac{1}{2}n+1)}{\tfrac{1}{2}\,.\,\tfrac{3}{2}\,.\,2\,!}u^2 + \ldots\right\}$$

$$= c_0\mathrm{F}(\tfrac{1}{2}n, \ -\tfrac{1}{2}n\,;\ \tfrac{1}{2}\,;\ u). \quad (8)$$

CASE II.—Let $\rho = \frac{1}{2}$; then

$$c_{\nu+1} = c_\nu \frac{(\frac{1}{2}+\nu)^2 - \frac{1}{4}n^2}{(\nu+\frac{3}{2})(\nu+1)} = c_\nu \frac{\left(\frac{1+n}{2}+\nu\right)\left(\frac{1-n}{2}+\nu\right)}{(\frac{3}{2}+\nu)(\nu+1)},$$

so that

$$y = c_0 u^{\frac{1}{2}}\left\{1 + \frac{\left(\frac{1+n}{2}\right)\left(\frac{1-n}{2}\right)}{\frac{3}{2}\,.\,1}u + \frac{\left(\frac{1+n}{2}\right)\left(\frac{1+n}{2}+1\right)\left(\frac{1-n}{2}\right)\left(\frac{1-n}{2}+1\right)}{\frac{3}{2}\,.\,\frac{5}{2}\,.\,2\,!}u^2 + \dots\right\}$$

$$= c_0 u^{\frac{1}{2}} \mathrm{F}\left(\frac{1+n}{2},\ \frac{1-n}{2};\ \frac{3}{2};\ u\right). \quad (9)$$

As the series in (8) and (9) both converge absolutely for $-1 < u < 1$, the solutions (8) and (9) are valid for that range.

Now these solutions are linearly independent. They therefore give two linearly independent solutions of (2). As $u = \sin^2 x$, they can be expanded in powers of $x$; this is also true for $\sin nx$ and $\cos nx$. Hence

$$\sin nx = \mathrm{C}\,\mathrm{F}(\tfrac{1}{2}n, -\tfrac{1}{2}n;\ \tfrac{1}{2};\ \sin^2 x) + \mathrm{D}\sin x\,\mathrm{F}\left(\frac{1+n}{2},\ \frac{1-n}{2};\ \frac{3}{2};\ \sin^2 x\right).$$

In order to find the values of C and D we assume that $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, so fixing a range of $x$ for which $|u| < 1$. On putting $x = 0$ it is seen that $\mathrm{C} = 0$. In order to determine D, divide the equation by $\sin x$, and let $x$ tend to zero.

Thus $n = \mathrm{D}$, and therefore

$$\sin nx = n \sin x\,\mathrm{F}\left(\frac{1+n}{2},\ \frac{1-n}{2};\ \frac{3}{2};\ \sin^2 x\right), \quad (10)$$

where $-\frac{1}{2}\pi \leqq x \leqq \frac{1}{2}\pi$, as the series converges for $\sin^2 x = 1$.

Again

$$\cos nx = \mathrm{K}\,\mathrm{F}(\tfrac{1}{2}n,\ -\tfrac{1}{2}n;\ \tfrac{1}{2};\ \sin^2 x) + \mathrm{L}\sin x\,\mathrm{F}\left(\frac{1+n}{2},\ \frac{1-n}{2};\ \tfrac{3}{2};\ \sin^2 x\right),$$

where K and L are constants. To determine these put $x = 0$ and get $1 = \mathrm{K}$. Next assume that $-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi$ and differentiate the equation. Thus

$$-n\sin nx = 2\sin x\cos x \times \{\text{a convergent series in } \sin^2 x\}$$
$$+ \mathrm{L}\cos x\,\mathrm{F}\left(\frac{1+n}{2},\ \frac{1-n}{2};\ \tfrac{3}{2};\ \sin^2 x\right)$$
$$+ \mathrm{L}\,.\,2\sin^2 x\cos x \times \{\text{a convergent series in } \sin^2 x\},$$

from which, on putting $x = 0$, we find that $\mathrm{L} = 0$. Hence

$$\cos nx = \mathrm{F}(\tfrac{1}{2}n,\ -\tfrac{1}{2}n;\ \tfrac{1}{2};\ \sin^2 x),\ -\tfrac{1}{2}\pi \leqq x \leqq \tfrac{1}{2}\pi. \quad (11)$$

On differentiating (10) and (11) term by term, it is found that

$$\cos nx = \cos x\,\mathrm{F}\left(\frac{1+n}{2},\ \frac{1-n}{2};\ \tfrac{1}{2};\ \sin^2 x\right),$$
$$-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi, \quad (12)$$

and

$$\sin nx = n\sin x\cos x\,\mathrm{F}(1+\tfrac{1}{2}n,\ 1-\tfrac{1}{2}n;\ \tfrac{3}{2};\ \sin^2 x),$$
$$-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi. \quad (13)$$

*Example* 1.—By equating the coefficients of $n$ and $n^2$ in formulæ (10) and (11) respectively, show that, if

$$-\tfrac{1}{2}\pi \leqq x \leqq \tfrac{1}{2}\pi,$$

(i) $x = \sin x + \frac{1}{2}\frac{\sin^3 x}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{\sin^5 x}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\sin^7 x}{7} + \ldots,$

(ii) $\frac{1}{2}x^2 = \frac{1}{2\,!}\sin^2 x + \frac{2^2}{4\,!}\sin^4 x + \frac{2^2\,.\,4^2}{6\,!}\sin^6 x + \ldots.$

[*Cf.* § 3, Examples 7, 8.]

*Example* 2.—Show that

(i) $\sin(\frac{1}{2}\pi x) = x\,\mathrm{F}\left(\frac{1+x}{2}, \frac{1-x}{2}\,;\ \frac{3}{2}\,;\ 1\right)$,

(ii) $\cos(\frac{1}{2}\pi x) = \mathrm{F}(\frac{1}{2}x, -\frac{1}{2}x\,;\ \frac{1}{2}\,;\ 1)$,

(iii) $\cos(\frac{1}{3}\pi x) = \mathrm{F}\left(x, -x\,;\ \frac{1}{2}\,;\ \frac{1}{2^2}\right)$,

(iv) $\sin(\frac{1}{3}\pi x) = \frac{\sqrt{3}}{2}x\,\mathrm{F}\left(1+x, 1-x\,;\ \frac{3}{2}\,;\ \frac{1}{2^2}\right)$,

(v) $$\frac{\pi^2}{48} = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1\,.\,3}{2\,.\,4}\left(1+\frac{1}{3^2}\right) + \frac{1}{7}\cdot\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)+\cdots$$

[For (i) and (ii) put $\frac{1}{2}\pi$ for $x$ and $x$ for $n$ in (10) and (11); for (iii) and (iv) put $\frac{1}{6}\pi$ for $x$ and $2x$ for $n$ in (11) and (13); for (v) equate the coefficients of $x^3$ in (i).]

## EXAMPLES XIX

1. Show that the series $\sum_{n=0}^{\infty} x(1-x)^n$ is convergent, but not uniformly convergent, for $0 \leqq x < 2$. Is there any interval of uniform convergence? Show that the sum is discontinuous at the origin, but that term by term integration in $(0, 1)$ leads to a correct result.

Ans. For $x = 0$ the sum is 0; for $0 < x < 2$ the sum is 1. The series converges uniformly for $\alpha \leqq x \leqq 2 - \alpha$, where $0 < \alpha < 1$.

2. Show that the series

$$\sum_{n=0}^{\infty} \frac{x}{(1+nx)\{1+(n+1)x\}}$$

is convergent for all values of $x$, but that it does not converge uniformly in the interval $0 \leqq x \leqq 1$. Verify that the integral of the sum of the series over that interval is equal to the sum of the integrals of the terms.

Show that term by term differentiation of the series is permissible except at $x = 0$.

3. Show that the series $\sum_{n=1}^{\infty} x^n(1-x^n)$ is convergent, but not uniformly convergent, in the closed interval (0, 1).

4. Show that the series

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}, \quad \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$$

are absolutely and uniformly convergent for all values of $\theta$.

5. If $p > 1$ and $\alpha$ is a constant, show that the series

$$\sum_{n=1}^{\infty} \frac{\sin n\alpha}{n^p} \frac{x^n}{1+x^{2n}}$$

converges uniformly for all values of $x$.

6. Show that the series

$$\sum_{n=2}^{\infty} \frac{\log n}{n} \cos n\theta$$

is uniformly convergent for $\alpha \leqq \theta \leqq 2\pi - \alpha$, where $0 < \alpha < \pi$.

7. Show that the series

$$\sum_{n=1}^{\infty} \frac{\theta(2\pi - \theta)\sin n\theta}{\sqrt{n}}$$

is uniformly convergent for $0 \leqq \theta \leqq 2\pi$.

8. Show that the series

$$\sum_{n=1}^{\infty} \frac{x^n(1-x)}{n^\alpha(1-x^n)}$$

is absolutely and uniformly convergent in the range

$$0 \leqq x < 1, \quad \text{if } \alpha > 0.$$

9. Show that the sum of the series

$$xe^{-\frac{1}{2}x^2} + x\{2e^{-x^2} - e^{-\frac{1}{2}x^2}\} + x\{3e^{-\frac{3}{2}x^2} - 2e^{-x^2}\} + \dots$$
$$\dots + x\left\{ne^{-\frac{n}{2}x^2} - (n-1)e^{-\frac{n-1}{2}x^2}\right\} + \dots$$

is continuous for all values of $x$, but that the series does not converge uniformly in any interval containing $x = 0$.

[$S_n = nxe^{-\frac{1}{2}nx^2}$. If $x \neq 0$, $S_n \to 0$ when $n \to \infty$; if $x = 0$, $S_n = 0$. Hence the sum S has the value zero for all values of $x$. Again, let $x = 1/\sqrt{n}$, so that $x$ lies in the interval (0, 1); then $S_n = \sqrt{n} \cdot e^{-\frac{1}{2}}$, and therefore, when $n$ tends to infinity, $S_n$ tends to infinity. Thus no $m$ can be found such that, in the closed interval (0, 1), $| R_n(x) | = S_n < \epsilon$ for $n \geqq m$.]

10. If $0 < k < 1$, show that

(i) $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1 - k^2 \sin^2 \theta)}} = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ k^2),$$

(ii) $$\int_0^{\frac{\pi}{2}} \sqrt{(1 - k^2 \sin^2 \theta)} d\theta = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ k^2).$$

11. If $f(x)$ denotes the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{nx^{n+1}}{(n+1)(n+2)},$$

show that the series obtained by differentiation is uniformly convergent in the range given by $| x | \leqq a < 1$. Prove that

$$\int_0^x \frac{f'(x)}{x} dx = \frac{1}{x^2} \{\log (1 + x) - x + \tfrac{1}{2}x^2\},$$

and then verify that

$$f(x) = \left(1 + \frac{2}{x}\right) \log (1 + x) - 2.$$

12. If $| r | < 1$, show that

$$\int_0^{2\pi} \frac{\sin^4 \theta \, d\theta}{1 - 2r \cos \theta + r^2} = \frac{\pi}{4}(3 - r^2).$$

13. If $| r | < 1$, prove that

$$r \cos \theta - \tfrac{1}{3}r^3 \cos 3\theta + \tfrac{1}{5}r^5 \cos 5\theta - \ldots = \tfrac{1}{2} \tan^{-1}\left(\frac{2r \cos \theta}{1 - r^2}\right).$$

14. If $-\frac{1}{4}\pi \leqq \theta \leqq \frac{1}{4}\pi$, show that

$$\log \cos \theta = -\tfrac{1}{2} \tan^2 \theta + \tfrac{1}{4} \tan^4 \theta - \tfrac{1}{6} \tan^6 \theta + \ldots.$$

15. Show that, if $n$ is a positive integer, and if $| a | < 1$,

(i) $$\int_0^{\pi} \log (1 - 2a \cos x + a^2) \cos nx \, dx = -\frac{\pi a^n}{n},$$

(ii) $$\int_0^{\pi} \frac{\cos nx \, dx}{1 - 2a \cos x + a^2} = \frac{\pi a^n}{1 - a^2}.$$

16. If $y = \sqrt{(1-x^2)}\sin^{-1}x$, show that

$$(1-x^2)\,Dy + xy = 1 - x^2,$$

and hence prove that, if $-1 \leqq x \leqq 1$,

$$\sqrt{(1-x^2)}\sin^{-1}x = x - \frac{x^3}{3} - \frac{2}{3}\frac{x^5}{5} - \frac{2\,.\,4}{3\,.\,5}\frac{x^7}{7} - \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\frac{x^9}{9} - \dots$$

17. If $-1 \leqq x \leqq 1$, show that

$$\log\{x + \sqrt{(1+x^2)}\} = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^7}{7} + \dots$$

18. If $-1 \leqq x \leqq 1$, show that

$$\log[\tfrac{1}{2}\{1 + \sqrt{(1-x)}\}] = -\frac{1}{2}\frac{x}{2} - \frac{1\,.\,3}{2\,.\,4}\frac{x^2}{4} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^3}{6} - \dots$$

Deduce that, if $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$,

$$2\log(\sec\tfrac{1}{2}\theta) = \frac{1}{2}\frac{\sin^2\theta}{2} + \frac{1\,.\,3}{2\,.\,4}\frac{\sin^4\theta}{4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\sin^6\theta}{6} + \dots$$

19. If $-\frac{1}{4}\pi \leqq \tan^{-1}x \leqq \frac{1}{4}\pi$, show that

$$(\tan^{-1}x)^2 = x^2 - (1 + \tfrac{1}{3})\frac{x^4}{2} + (1 + \tfrac{1}{3} + \tfrac{1}{5})\frac{x^6}{3} - \dots .$$

20. If $f(x) = \cos(a\sin^{-1}x)$, show that

$$(1-x^2)f''(x) - xf'(x) + a^2f(x) = 0,$$

and then prove that, if $-1 \leqq x \leqq 1$,

$$\cos(a\sin^{-1}x) = 1 - \frac{a^2x^2}{2\,!} + \frac{a^2(a^2-2^2)x^4}{4\,!} - \frac{a^2(a^2-2^2)(a^2-4^2)x^6}{6\,!} + \dots .$$

Deduce that

$$\cos\left(\frac{\pi}{3}\theta\right) = 1 - \frac{\theta^2}{2\,!} + \frac{\theta^2(\theta^2-1^2)}{4\,!} - \frac{\theta^2(\theta^2-1^2)(\theta^2-2^2)}{6\,!} + \dots .$$

21. Show that, if $-1 \leqq x \leqq 1$,

$$\sin(a\sin^{-1}x) = ax - \frac{a(a^2-1^2)x^3}{3\,!} + \frac{a(a^2-1^2)(a^2-3^2)x^5}{5\,!} - \dots .$$

22. If $$f(x) = \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \dots,$$
show that, if $-1 \leqq x < 1$,
$$x^2 f'(x) = -x - \log(1 - x),$$
and deduce that
$$f(x) = 1 + \frac{(1 - x)\log(1 - x)}{x}.$$

23. Show that, if $-1 \leqq x \leqq 1$,
$$\frac{\log\{x + \sqrt{(1 + x^2)}\}}{\sqrt{(1 + x^2)}} = x - \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 - \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}x^7 + \dots.$$

24. Show that, if $-1 \leqq x \leqq 1$,
$$\tfrac{1}{2}[\log\{\sqrt{(x^2 + 1)} - x\}]^2 = \frac{x^2}{2!} - \frac{2^2x^4}{4!} + \frac{2^2 \cdot 4^2x^6}{6!} - \frac{2^2 \cdot 4^2 \cdot 6^2x^8}{8!} + \dots.$$

25. If $\sum_{n=1}^{\infty} u_n$ is convergent, prove that the series $\sum_{n=1}^{\infty} u_n v_n(x)$ is uniformly convergent in an interval $(a, b)$ when, for any value of $x$ in this interval, $v_n(x)$ is positive and never increases with $n$, and $v_1(x)$ is less than a fixed quantity $k$.

Hence show that the following series are uniformly convergent in the interval $(0, 1)$:

(i) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n(1 + x^{2n})},$$

(ii) $$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)\frac{(-1)^{n-1}x^n}{n + 1}.$$

26. If $$f(x) = \sum_{n=0}^{\infty} \left\{\frac{x^{2n+1}}{2n + 1} - \frac{x^{2n+2}}{2n + 2}\right\},$$

and $$\phi(x) = \sum_{n=0}^{\infty} \left\{\frac{x^{2n+1}}{2n + 1} - \frac{x^{n+1}}{2n + 2}\right\},$$

show that $f(x)$ is continuous for $-1 < x \leq 1$, while $\phi(x)$ is continuous for $-1 < x < 1$, but has a finite discontinuity at $x = 1$. Explain this discrepancy.

Ans. If $|x| < 1$, $f(x) = \log(1+x)$, $\phi(x) = \frac{1}{2}\log(1+x)$; but when $x = 1$, $f(1) = \phi(1) = \log 2$. In the proof of Abel's Theorem it is assumed that the series is arranged in ascending powers of $x$. The theorem does not, therefore, apply to $\phi(x)$.

27. If
$$f(x) = \sum_{n=0}^{\infty}\left\{\frac{x^{n+1}}{n+1} - \frac{2x^{2n+3}}{2n+3}\right\},$$

show that $f(x)$ is continuous for $-1 < x < 1$, but is discontinuous at $x = 1$.

Ans. If $|x| < 1$, $f(x) = 2x - \log(1+x)$, but $f(1) = 2 - 2\log 2$.

28. Prove that

$$-2^8 \cos^3\theta \sin^6\theta = \cos 9\theta - 3\cos 7\theta + 8\cos 3\theta - 6\cos\theta,$$
and *deduce* that

$$9^4 - 3 \cdot 7^4 + 8 \cdot 3^4 - 6 = 0,$$
$$9^6 - 3 \cdot 7^6 + 8 \cdot 3^6 - 6 = 2^8 \cdot 6\,!$$

29. If $n$ is an odd integer, show that

$$\sum_{k=1}^{\frac{n-1}{2}} \operatorname{cosec}^2 \frac{k\pi}{n} = \frac{n^2-1}{6}.$$

[In formula (26), Chapter XV, equate the coefficients of $\theta^3$.]

30. If $y = e^{b\sin^{-1}x} = \sum_{n=0}^{\infty} a_n x^n$, show that

$$(1-x^2)D^2y - xDy - b^2y = 0,$$

and deduce that

$$(n+1)(n+2)a_{n+2} = (n^2+b^2)a_n.$$

Hence obtain the expansion $(-1 \leq x \leq 1)$

$$\tfrac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2\,!} + \frac{2^2 \cdot x^4}{4\,!} + \frac{2^2 \cdot 4^2 \cdot x^6}{6\,!} + \ldots.$$

31. Transform the equation $\frac{d^2y}{dx^2} = n^2y$ by means of the substitution $u = \cosh^2 x$. Hence show that, if $x$ and $n$ are positive,

$$e^{-nx} = (2\cosh x)^{-n}F\left(\frac{n}{2}, \frac{1+n}{2}; 1+n; \frac{1}{\cosh^2 x}\right).$$

32. The equation

$$x\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + 3y = 0$$

has a solution of the form $y = \sum_{n=0}^{\infty} A_n x^n$; find the values of the constants $A_1, A_2, \ldots$ in terms of $A_0$, and state the range of $x$ for which the series is convergent.

Ans. $A_n/A_0 = (-1)^n\frac{1}{2}(n+2)/n!$ The series converges for all values of $x$.

33. Prove that $v_1e^{-\frac{1}{4}x^2}$, $v_2e^{-\frac{1}{4}x^2}$, where

$$v_1 = 1 - \frac{n}{2!}x^2 + \frac{n(n-2)}{4!}x^4 - \frac{n(n-2)(n-4)}{6!}x^6 + \ldots,$$

$$v_2 = x - \frac{n-1}{3!}x^3 + \frac{(n-1)(n-3)}{5!}x^5 - \ldots,$$

are solutions of the equation

$$y'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 0.$$

CHAPTER XX

# INFINITE PRODUCTS : FUNCTIONS OF A COMPLEX VARIABLE

## § 1. Tannery's Theorem

THIS important theorem, by means of which expansions in infinite series can be derived from known expansions containing only a finite number of terms, can be stated as follows :

Let
$$F(n) \equiv \sum_{r=0}^{N} u_r(n),$$

where $u_r(n)$ is a function of $n$, and N is either infinity or a function of $n$ which tends to infinity with $n$. If the following conditions are fulfilled :

(i) $u_r(n)$ tends to a definite limit $v_r$ when $n$ tends to infinity, for every value of $r$,

(ii) $|u_r(n)| \leqq M_r$, where $M_r$ is a (positive) number independent of $n$, for every $r$,

(iii) the series $\sum_{r=0}^{\infty} M_r$ is convergent,

then, when $n$ tends to infinity, $F(n)$ tends to the limit $\sum_{r=0}^{\infty} v_r$.

Since $|u_r(n)| \leqq M_r$ and $u_r(n)$ tends to $v_r$ when $n$ tends to infinity, $|v_r| \leqq M_r$. Thus the series $\Sigma v_r$ is absolutely convergent.

Now, given $\epsilon$, choose $m$, a positive integer, so large that

$$\sum_{r=m+1}^{\infty} M_r < \tfrac{1}{3}\epsilon,$$

and let $n$ be so large that $N > m$. Write

$$F(n) - \sum_{r=0}^{\infty} v_r = \alpha + \beta + \gamma,$$

where
$$\alpha = \sum_{r=0}^{m} \{u_r(n) - v_r\},$$

$$\beta = \sum_{r=m+1}^{N} u_r(n),$$

$$\gamma = - \sum_{r=m+1}^{\infty} v_r.$$

Then

$$|\beta| \leqq \sum_{r=m+1}^{N} M_r \leqq \sum_{r=m+1}^{\infty} M_r < \tfrac{1}{3}\epsilon,$$

and
$$|\gamma| \leqq \sum_{r=m+1}^{\infty} M_r < \tfrac{1}{3}\epsilon.$$

Now the value of $m$ depends only on the series $\Sigma M_r$, and is therefore independent of $n$. Having chosen $m$ so that the inequality $\sum_{r=m+1}^{\infty} M_r < \frac{1}{3}\epsilon$ is satisfied, keep it fixed. The series $\alpha$ contains only $m+1$ terms, and therefore a number $n_1$ can be found so large that, if $n \geqq n_1$, $|\alpha| < \frac{1}{3}\epsilon$. Thus, if $n \geqq n_1$, $n_1$ being chosen so large that $N > m$,

$$|F(n) - \sum_{r=0}^{\infty} v_r| \leqq |\alpha| + |\beta| + |\gamma| < \epsilon.$$

Hence it follows that, when $n$ tends to infinity, $F(n)$ tends to $\sum_{r=0}^{\infty} v_r$.

*Example.*—From the expansions [Ch. XV, § 4 (5a), (6a)]

(a) $\cos n\theta = \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta + \ldots,$

(b) $\sin n\theta = \frac{n}{1!} \cos^{n-1} \theta \sin \theta$

$$- \frac{n(n-1)(n-2)}{3!} \cos^{n-3} \theta \sin^3 \theta + \ldots,$$

where $n$ is a positive integer, deduce the expansions

(c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

[In (a) put $n\theta = x$. Then

$$u_r = (-1)^r \frac{x^{2r}}{(2r)!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots$$

$$\ldots \left(1 - \frac{2r-1}{n}\right)\left(\cos \frac{x}{n}\right)^{n-2r} \left(\frac{\sin \frac{x}{n}}{\frac{x}{n}}\right)^{2r},$$

where $2r \leqq n$.

Now $$\mathop{\mathcal{L}}_{n\to\infty} \left(\frac{\sin \frac{x}{n}}{\frac{x}{n}}\right)^{2r} = 1$$

and (Ch. XVII, § 6, Example 1)

$$\mathop{\mathcal{L}}_{n\to\infty} \left(\cos \frac{x}{n}\right)^{n-2r} = 1.$$

Therefore

$$\mathop{\mathcal{L}}_{n\to\infty} u_r = (-1)^r \frac{x^{2r}}{(2r)!}.$$

Also

$$| u_r | \leqq \frac{| x^{2r} |}{(2r)!}.$$

But $\Sigma\{x^{2r}/(2r)!\}$ is absolutely convergent. Hence, by Tannery's Theorem, (c) follows from (a). In the same way (d) can be deduced from (b).]

## § 2. Expansion of cot $\theta$ in an Infinite Series of Partial Fractions

If $n$ is a positive integer

$$\cos\theta + i\sin\theta = \left(\cos\frac{\theta}{2n} + i\sin\frac{\theta}{2n}\right)^{2n},$$

whence, on equating imaginary parts, we deduce that

$$\sin\theta = {}^{2n}C_1\left(\cos\frac{\theta}{2n}\right)^{2n-1}\sin\frac{\theta}{2n} - {}^{2n}C_3\left(\cos\frac{\theta}{2n}\right)^{2n-3}\left(\sin\frac{\theta}{2n}\right)^3 + \ldots \ldots + (-1)^{n-1}\,{}^{2n}C_{2n-1}\cos\frac{\theta}{2n}\left(\sin\frac{\theta}{2n}\right)^{2n-1}$$

From the substitution

$$\cos^2\frac{\theta}{2n} = 1 - \sin^2\frac{\theta}{2n},$$

it follows that

$$\sin\theta = \sin\frac{\theta}{2n}\cos\frac{\theta}{2n}\left\{A_0 + A_2\sin^2\frac{\theta}{2n} + \ldots + A_{2n-2}\left(\sin\frac{\theta}{2n}\right)^{2n-2}\right\}$$

But $\sin\theta$ vanishes when $\theta = r\pi$, where $r = 0, \pm 1, \pm 2, \ldots$; hence

$$\sin\theta = A\sin\frac{\theta}{n}\prod_{r=1}^{n-1}\left(1 - \frac{\sin^2\frac{\theta}{2n}}{\sin^2\frac{r\pi}{2n}}\right).$$

On dividing by $\sin(\theta/n)$, and making $\theta$ tend to zero, we find that $A = n$; thus

$$\sin\theta = n\sin\frac{\theta}{n}\prod_{r=1}^{n-1}\left(1-\frac{\sin^2\dfrac{\theta}{2n}}{\sin^2\dfrac{r\pi}{2n}}\right).$$

Next, differentiate this equation logarithmically, and get

$$\cot\theta = \frac{1}{n}\cot\frac{\theta}{n} + \frac{1}{n}\sum_{r=1}^{n-1}\frac{\sin\dfrac{\theta}{2n}\cos\dfrac{\theta}{2n}}{\sin^2\dfrac{\theta}{2n}-\sin^2\dfrac{r\pi}{2n}}, \quad (1)$$

where $\theta \neq k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Now let

$$u_r(n) = \frac{1}{n}\,\frac{\sin\dfrac{\theta}{2n}\cos\dfrac{\theta}{2n}}{\sin^2\dfrac{r\pi}{2n}-\sin^2\dfrac{\theta}{2n}}.$$

Then

$$u_r(n) = \frac{2}{\theta}\cos\frac{\theta}{2n}\left(\frac{\dfrac{\theta}{2n}}{\sin\dfrac{\theta}{2n}}\right)\frac{1}{\left(\dfrac{\sin\dfrac{r\pi}{2n}}{\dfrac{r\pi}{2n}}\right)^2\left(\dfrac{\dfrac{\theta}{2n}}{\sin\dfrac{\theta}{2n}}\right)^2\dfrac{r^2\pi^2}{\theta^2}-1},$$

and therefore

$$\mathop{\mathcal{L}}_{n\to\infty} u_r(n) = \frac{2\theta}{r^2\pi^2-\theta^2} \equiv v_r, \text{ say.}$$

Now let $p$ be a positive integer so large that $\left|\dfrac{\theta}{2p}\right| < \dfrac{\pi}{2}$ and therefore, if $n \geqq p$, $\left|\dfrac{\theta}{2n}\right| < \dfrac{\pi}{2}$. Then, if $n \geqq p$,

$$0 < \cos\frac{\theta}{2n} < 1, \qquad 1 < \frac{\dfrac{\theta}{2n}}{\sin\dfrac{\theta}{2n}} < \frac{\pi}{2}.^{*}$$

Also, since $r \leqq n-1$, it follows that $0 < r\pi/(2n) < \frac{1}{2}\pi$, and therefore

$$1 > \frac{\sin\dfrac{r\pi}{2n}}{\dfrac{r\pi}{2n}} > \frac{2}{\pi}.$$

Hence, if $m$ is a positive integer, chosen so that $m > \frac{1}{2}\,|\,\theta\,|$, and if $r \geqq m$, $n \geqq p$,

$$|\,u_r(n)\,| < \frac{2}{|\,\theta\,|}\cdot\frac{\pi}{2}\cdot\frac{1}{\left(\dfrac{2}{\pi}\right)^2\cdot\dfrac{r^2\pi^2}{\theta^2}-1} = \frac{\pi\,.\,|\,\theta\,|}{4r^2-\theta^2} \equiv \mathrm{M}_r,$$

say.

But the series $$\sum_{r=m}^{\infty}\mathrm{M}_r$$

is convergent. Hence, if

$$\mathrm{F}(n) = \sum_{r=m}^{n-1} u_r(n),$$

it follows from Tannery's Theorem that

$$\mathop{\mathcal{L}}_{n\to\infty}\mathrm{F}(n) = \sum_{r=m}^{\infty} v_r.$$

* If $0 < \theta < \frac{1}{2}\pi$, $1 < \theta/\sin\theta < \frac{1}{2}\pi$. For

$$\frac{d}{d\theta}\frac{\theta}{\sin\theta} = \frac{\cos\theta}{\sin^2\theta}(\tan\theta-\theta) > 0, \quad \text{if } 0 < \theta < \tfrac{1}{2}\pi.$$

Thus $\theta/\sin\theta$ increases as $\theta$ increases from 0 to $\frac{1}{2}\pi$.

But (1) can be written

$$\cot\theta = \frac{1}{n}\cot\frac{\theta}{n} - \sum_{r=1}^{m-1} u_r(n) - F(n).$$

Thus, when $n$ tends to infinity, we have

$$\cot\theta = \frac{1}{\theta} + \sum_{r=1}^{\infty} \frac{2\theta}{\theta^2 - r^2\pi^2}, \qquad (2)$$

where $\theta \neq k\pi$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

For an alternative proof of (2) see § 4, Note 1.

*Example* 1.—Prove that, if $\theta \neq n\pi$, where

$$n = 0, \pm 1, \pm 2, \ldots,$$

$$\cot\theta = \frac{1}{\theta} + \sum_{n=1}^{\infty}\left(\frac{1}{\theta - n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\theta + n\pi} - \frac{1}{n\pi}\right).$$

[From (2),

$$\cot\theta = \frac{1}{\theta} + \mathop{\mathcal{L}}_{p\to\infty} \sum_{n=1}^{p}\left(\frac{1}{\theta - n\pi} + \frac{1}{\theta + n\pi}\right)$$

$$= \frac{1}{\theta} + \mathop{\mathcal{L}}_{p\to\infty}\left\{\sum_{n=1}^{p}\left(\frac{1}{\theta - n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{p}\left(\frac{1}{\theta + n\pi} - \frac{1}{n\pi}\right)\right\},$$

from which the result follows.]

*Example* 2.—Show that, if $\theta \neq n\pi$, where

$$n = 0, \pm 1, \pm 2, \ldots,$$

$$\operatorname{cosec}\theta = \frac{1}{\theta} + \sum_{n=1}^{\infty}(-1)^n \frac{2\theta}{\theta^2 - n^2\pi^2}$$

$$= \frac{1}{\theta} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\theta - n\pi} + \frac{1}{n\pi}\right)$$

$$+ \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\theta + n\pi} - \frac{1}{n\pi}\right).$$

[Employ the identity $\operatorname{cosec}\theta = \cot\frac{1}{2}\theta - \cot\theta$.]

*Example* 3.—Show that, if $\theta \neq (n + \frac{1}{2})\pi$, where

$$n = 0, \pm 1, \pm 2, \ldots,$$

$$\text{(i)}\ \tan\theta = 8\sum_{n=0}^{\infty} \frac{\theta}{(2n+1)^2\pi^2 - 4\theta^2}$$

$$= \sum_{n=-\infty}^{\infty} \left\{\frac{1}{(n+\frac{1}{2})\pi - \theta} - \frac{1}{(n+\frac{1}{2})\pi}\right\},$$

$$\text{(ii)}\ \sec\theta = 4\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\pi}{(2n+1)^2\pi^2 - 4\theta^2}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \left\{\frac{1}{(n+\frac{1}{2})\pi - \theta} - \frac{1}{(n+\frac{1}{2})\pi}\right\}.$$

*Example* 4.—Prove that

$$\text{(i)}\ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6},$$

$$\text{(ii)}\ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

[For (i) multiply (2) by $\theta$, assume that $|\theta|$ is small, and expand both sides of the equation in powers of $\theta$, equating the coefficients of $\theta^2$. For (ii) multiply (i) by $\frac{1}{4}$ and subtract the resulting equation from (i).]

*Example* 5.—Prove that, if $x$ is not integral,

$$\pi^2 \operatorname{cosec}^2 \pi x = \sum_{n=-\infty}^{\infty} (x+n)^{-2}.$$

*Example* 6.—Prove that

$$\coth x = \frac{1}{x} + \sum_{r=1}^{\infty} \frac{2x}{x^2 + r^2\pi^2}.$$

$$\left[\frac{(1+x/n)^{n-1} + (1-x/n)^{n-1}}{(1+x/n)^n - (1-x/n)^n}\right.$$

$$= \frac{A}{x} + \sum_{r=1}^{N} \left\{\frac{A_r}{x - in\tan\frac{r\pi}{n}} + \frac{B_r}{x + in\tan\frac{r\pi}{n}}\right\},$$

where $N = \frac{1}{2}(n-1)$ or $\frac{1}{2}(n-2)$ according as $n$ is odd or even.

Here $A = A_r = B_r = 1$, so that R.H.S. $= \frac{1}{x} + \sum_{r=1}^{N} u_r$, where

$$u_r = 2x\cos^2\frac{r\pi}{n} \div \left\{r^2\pi^2\left(\sin\frac{r\pi}{n}\Big/\frac{r\pi}{n}\right)^2 + x^2\cos^2\frac{r\pi}{n}\right\},$$

and, if $r > \frac{1}{2}|x|$,

$$|u_r| < \frac{2|x|}{4r^2 - |x|^2}.$$

The result follows by Tannery's Theorem.]

*Example* 7.—Show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \tfrac{1}{2}(\pi\coth\pi - 1).$$

## § 3. Infinite Products

Let

$$P_n = \prod_{r=1}^{n} f_r,$$

where the symbol on the right denotes the product $f_1 f_2 \ldots f_n$. Then if, when $n$ tends to infinity, $P_n$ tends to a definite non-zero limit P, the infinite product

$$\prod_{r=1}^{\infty} f_r$$

is said to be *convergent* and to converge to the value P.

If $P_n$ tends to $+\infty$ or $-\infty$, the product is said to be *divergent*.

If $P_n$ tends to zero, the product is said to *diverge to zero.*

If, however, each of a finite number of factors has the value zero, the product is convergent if it converges when these factors are removed. In such cases the product has the *value* zero.

If $P_n$ does not tend to a definite limit or to $+\infty$ or $-\infty$, it is said to *oscillate*.

*Note.*—If the product is convergent, $f_n \to 1$ when $n \to \infty$. For $f_n = P_n/P_{n-1}$, and $P_n$ and $P_{n-1}$ both tend to the same non-zero limit P when $n \to \infty$.

*Example* 1.—$P_n = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1} = \frac{1}{n+1}$. Here $P_n \to 0$. The product diverges to zero.

*Example* 2.—$P_n = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} = n+1$. Here $P_n \to \infty$. The product is divergent.

*Example* 3.—$P_n = \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdots$ to $n$ factors. Here $P_n$ is equal to $\frac{1}{2}$ or 1 according as $n$ is odd or even. The product oscillates.

*Example* 4.—$P_n = (-1) \cdot (-1) \cdot (-1) \cdots$ to $n$ factors. The product oscillates.

*Example* 5.—$P_n = (-1) \cdot (-2) \cdot (-3) \cdots (-n)$. The product oscillates infinitely.

*Example* 6.—If $-1 < x < 1$, show that the product

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$$

converges to $1/(1-x)$.

*Example* 7.—Show that

(i) $\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots = \frac{\sin\theta}{\theta}$,

(ii) $\left(1-\frac{4}{3}\sin^2\frac{\theta}{3}\right)\left(1-\frac{4}{3}\sin^2\frac{\theta}{3^2}\right)\left(1-\frac{4}{3}\sin^2\frac{\theta}{3^3}\right)\cdots = \frac{\sin\theta}{\theta}$.

*Example* 8.—Show that

(i) $\prod_{n=2}^{\infty}\left\{1-\frac{2}{n(n+1)}\right\} = \frac{1}{3}$, (ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$.

Theorem.—If the series $\sum_{r=1}^{\infty}\log f_r$, where each $f_r$ is positive, converges to the sum S, the product $\prod_{r=1}^{\infty} f_r$ converges to the (non-zero) value $e^S$.

Let $\quad S_n = \log f_1 + \log f_2 + \ldots + \log f_n;$

then, since $\quad f_r = e^{\log f_r},$

$$P_n \equiv \prod_{r=1}^{n} f_r = e^{S_n}.$$

Now, when $n$ tends to infinity, $S_n$ tends to S, and therefore, as the exponential function is continuous, $e^{S_n}$ tends to $e^S$. Hence, when $n$ tends to infinity, $P_n$ tends to $e^S$.

If $S_n$ tends to $+\infty$, $P_n$ tends to infinity, while, if $S_n$ tends to $-\infty$, $P_n$ tends to zero. Thus, if the sequence $(S_n)$ is convergent, the product is convergent; if the sequence diverges to $+\infty$, the product diverges to $+\infty$; and if the sequence diverges to $-\infty$, the product diverges to zero.

*Note.*—The condition stated in the above theorem for the convergence of the product $\Pi f_r$, namely, that the product is convergent if the series $\Sigma \log f_r$ is convergent, is not merely a *sufficient* condition; it is a *necessary* condition. Since $f_n$ tends to 1 when $n$ tends to infinity, only a finite number of the factors can be negative (or zero). Omit these to ensure that $f_r$ is always positive, and that, in consequence, $\log f_r$ has always a real value. Then, since $P_n = e^{S_n}$, $S_n = \log P_n$. Therefore, if $P_n$ tends to a definite (positive) non-zero limit **P**, $S_n$ must tend to a definite limit $\log$ **P**.

The product $\Pi f_r$ is said to be *absolutely convergent* or *conditionally convergent* according as the series $\Sigma \log f_r$ is absolutely or conditionally convergent. If the product is absolutely convergent, its value is not altered if the order of the factors is altered: this follows from the corresponding theorem (Ch. XVIII, § 5) for absolutely convergent series. On the other hand, if the product is conditionally convergent, a derangement of the factors may alter its value or make it divergent.

*Example* 9.—Prove that, if the series $\Sigma u_n$ is absolutely convergent, the product $\Pi(1 + u_n)$ is absolutely convergent.

Choose $m$ so large that, if $n \geqq m$, $|u_n| < 1$.
Then

$$\log(1 + u_n) = u_n - \tfrac{1}{2}u_n^2 + \tfrac{1}{3}u_n^3 - \ldots,$$

and therefore

$$|\log(1 + u_n)| \leqq |u_n| + |u_n|^2 + |u_n|^3 + \ldots.$$

Now let $k$ be the greatest of the moduli

$$|u_m|, |u_{m+1}|, |u_{m+2}|, \ldots; \text{ then, if } n \geqq m,$$

$$|\log(1 + u_n)| \leqq |u_n| \times (1 + k + k^2 + \ldots)$$

$$\leqq \frac{|u_n|}{1 - k}.$$

Thus, by the Comparison Theorem, $\sum_{n=m}^{\infty} \log(1 + u_n)$ is absolutely convergent. It follows that the infinite product is absolutely convergent.

*Example* 10.—If $a_n$ is always positive, show that the infinite product $\Pi(1 + a_n)$ converges or diverges according as the infinite series $\Sigma a_n$ converges or diverges.

If $\Sigma a_n$ converges, the result follows from Example 9. If $\Sigma a_n$ diverges,

$$\prod_{r=1}^{n} (1 + a_r) \geqq \sum_{r=1}^{n} a_r.$$

But, when $n \to \infty$, the R.H.S. $\to \infty$: hence the L.H.S. also $\to \infty$.

*Example* 11.—If the series $\Sigma u_n^2$ is convergent, prove that the product

$$\prod_{n=1}^{\infty} \{(1 + u_n)e^{-u_n}\}$$

is absolutely convergent.

Choose $m$ so large that, for $n \geqq m$, $|u_n| < 1$. Then, as in Example 9,

$$|\log(1 + u_n) - u_n| \leqq \frac{u_n^2}{1 - k},$$

and the result follows by the Comparison Test.

*Example* 12.—Show that the product

$$\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right\}$$

is absolutely convergent.

*Example* 13.—If the series $\Sigma u_n{}^2$ is convergent, show that the infinite product $\Pi(1+u_n)$ converges, diverges, or diverges to zero, according as $\Sigma u_n$ converges, diverges to $+\infty$, or diverges to $-\infty$.

From Example 11 it follows that the series

$$\Sigma\,\{\log(1+u_n)-u_n\}$$

is convergent. If its sum is L, and if the sums to $n$ terms of the series $\Sigma\log(1+u_n)$ and $\Sigma u_n$ are $S_n$ and $\Sigma_n$ respectively, then

$$\mathop{\mathcal{L}}_{n\to\infty} S_n = \mathop{\mathcal{L}}_{n\to\infty}\Sigma_n + L.$$

From this the result follows.

*Example* 14.—Prove that the products

$$\prod_{n=1}^{\infty}\left\{1+(-1)^{n-1}\frac{1}{n}\right\},\quad \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right),\quad \prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$$

are conditionally convergent, divergent and divergent to zero respectively.

*Example* 15.—Show that the product

$$\prod_{n=2}^{\infty}\left\{1-\frac{(-1)^n}{\sqrt{n}}\right\}$$

diverges to zero.

[Show that $\sum\left\{\log\left(1-\frac{(-1)^n}{\sqrt{n}}\right)+\left(\frac{(-1)^n}{\sqrt{n}}+\frac{1}{2n}\right)\right\}$ is absolutely convergent, and that therefore $\sum\log\left(1-\frac{(-1)^n}{\sqrt{n}}\right)$ diverges to $-\infty$.]

**Tannery's Theorem for Products.**

If, with the notation of § 1,

$$\mathrm{P}(n) \equiv \prod_{r=0}^{\mathrm{N}} \{1 + u_r(n)\},$$

and if the conditions (i), (ii), (iii) of § 1 are fulfilled, P($n$) tends to the limit

$$\prod_{r=0}^{\infty} (1 + v_r),$$

when $n$ tends to infinity.

Choose a positive integer $m$ so large that, if $r \geqq m$, $\mathrm{M}_r < 1$, and therefore $| u_r(n) | < 1$. Then, if $r \geqq m$,

$$\log \{1 + u_r(n)\} = u_r(n) - \tfrac{1}{2}\{u_r(n)\}^2 + \tfrac{1}{3}\{u_r(n)\}^3 - \ldots,$$

and therefore

$$| \log \{1 + u_r(n)\} | \leqq \mathrm{M}_r + \mathrm{M}_r^2 + \mathrm{M}_r^3 + \ldots = \frac{\mathrm{M}_r}{1 - \mathrm{M}_r}.$$

Thus

$$| \log \{1 + u_r(n)\} | \leqq \frac{\mathrm{M}_r}{1 - \mu},$$

where $\mu$ is the greatest of $\mathrm{M}_m$, $\mathrm{M}_{m+1}$, . . . , and is therefore independent of $r$ and $n$.

Hence, by Tannery's Theorem for series, the sum

$$\sum_{r=m}^{\mathrm{N}} \log \{1 + u_r(n)\}$$

tends to the value

$$\sum_{r=m}^{\infty} \log (1 + v_r),$$

when $n$ tends to infinity.

Thus the product

$$\prod_{r=m}^{N} \{1 + u_r(n)\}$$

tends to

$$\prod_{r=m}^{\infty} (1 + v_r),$$

and from this the theorem follows.

## § 4. Infinite Products for $\sin\theta$ and $\cos\theta$

If $p\pi < \theta < (p+1)\pi$, where $p$ is zero or a positive integer, then, from (2),

$$\int_0^\theta \left\{\cot\theta - \frac{1}{\theta} - \sum_{r=1}^{p} \frac{2\theta}{\theta^2 - r^2\pi^2}\right\} d\theta = \sum_{r=p+1}^{\infty} \int_0^\theta \frac{-2\theta d\theta}{r^2\pi^2 - \theta^2},$$

[See Ch. XIX, § 2, Example 3.] Hence

$$\log\left\{\frac{(-1)^p \sin\theta}{\theta \prod_{r=1}^{p}\left(\frac{\theta^2}{r^2\pi^2} - 1\right)}\right\} = \sum_{r=p+1}^{\infty} \log\left(1 - \frac{\theta^2}{r^2\pi^2}\right).$$

Thus

$$\frac{(-1)^p \sin\theta}{\theta \prod_{r=1}^{p}\left(\frac{\theta^2}{r^2\pi^2} - 1\right)} = \prod_{r=p+1}^{\infty}\left(1 - \frac{\theta^2}{r^2\pi^2}\right),$$

and therefore

$$\sin\theta = \theta \prod_{r=1}^{\infty}\left(1 - \frac{\theta^2}{r^2\pi^2}\right). \qquad (3)$$

As the functions on both sides of the equation are odd, the formula holds for negative as well as for positive values of $\theta$.

Again

$$\frac{\sin 2\theta}{2\sin\theta} = \frac{\mathcal{L}_{n\to\infty}\ 2\theta \prod_{r=1}^{2n}\left(1-\frac{4\theta^2}{r^2\pi^2}\right)}{\mathcal{L}_{n\to\infty}\ 2\theta \prod_{r=1}^{n}\left(1-\frac{4\theta^2}{4r^2\pi^2}\right)} =$$

and therefore

$$\mathcal{L}_{n\to\infty} \prod_{r=1}^{n}\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\},$$

*Alternative Proof.*—The formula (3) may be deduced from the formula

$$\sin\theta = n\sin\frac{\theta}{n}\prod_{r=1}^{n-1}\left(1-\sin^2\frac{\theta}{2n}\Big/\sin^2\frac{r\pi}{2n}\right)$$

(see § 2) by applying Tannery's Theorem for products For, if $p$ is a positive integer chosen so large that

$$\left|\frac{\theta}{2p}\right| < \frac{\pi}{2},$$

then, as in § 2, if $n \geqq p$,

$$\sin^2\frac{\theta}{2n}\Big/\sin^2\frac{r\pi}{2n} < \left(\frac{\theta}{2n}\right)^2\Big/\left(\frac{r}{n}\right)^2 = \frac{\theta^2}{4r^2} \equiv M_r,$$

and, consequently, the theorem is applicable.

*Note* 1.—Formula (2) can now be derived from formula (3) by logarithmic differentiation (see Ch. XIX, § 2, Example 3).

*Example* 1.—Show that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ .\ .\ .\ = \frac{\pi^4}{90}.$$

If $x$ is small,

$$\log\frac{\sin x}{x} = \log\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\ .\ .\ .\right) = \sum_{n=1}^{\infty}\log\left(1-\frac{x^2}{n^2\pi^2}\right).$$

Expand both sides in powers of $x$, and equate the coefficients of $x^4$.

*Example* 2.—Show that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}.$$

*Note* 2.—As the infinite series $\Sigma\{\theta/(r\pi)\}$ and $\Sigma\{-\theta/(r\pi)\}$ are not convergent, it is not permissible (§ 3, Example 13) to write the product (3) in the form

$$\sin\theta = \theta \prod_{r=-\infty}^{\infty}{}' \left(1 - \frac{\theta}{r\pi}\right),$$

where the dash indicates that there is no factor corresponding to $r = 0$. The formula can, however, be written

$$\sin\theta = \theta \mathop{\mathcal{L}}_{n\to\infty} \prod_{r=-n}^{n}{}' \left(1 - \frac{\theta}{r\pi}\right), \quad . \quad . \quad (4)$$

where the product must be put in convergent form before it can be written as an infinite product.

*Example* 3.—Derive the formula $\sin(\theta + \pi) = -\sin\theta$ from the product formula for $\sin\theta$.

From (4)

$$\sin(\theta + \pi) = (\theta + \pi) \mathop{\mathcal{L}}_{n\to\infty} \prod_{r=-n}^{n}{}' \left(1 - \frac{\theta + \pi}{r\pi}\right)$$

$$= (\theta + \pi) \mathop{\mathcal{L}}_{n\to\infty} \prod_{r=1}^{n} \left\{\frac{(r-1)\pi - \theta}{r\pi} \cdot \frac{(r+1)\pi + \theta}{r\pi}\right\}$$

$$= -\theta \mathop{\mathcal{L}}_{n\to\infty} \left[\prod_{r=1}^{n} \left\{\frac{r\pi - \theta}{r\pi} \cdot \frac{r\pi + \theta}{r\pi}\right\} \frac{(n+1)\pi + \theta}{n\pi - \theta}\right]$$

$$= -\theta \prod_{r=1}^{\infty} \left(1 - \frac{\theta^2}{r^2\pi^2}\right) = -\sin\theta.$$

*Example* 4.—Show that

$$\sin\theta = \theta \prod_{n=-\infty}^{\infty}{}' \left\{\left(1-\frac{\theta}{n\pi}\right)e^{\frac{\theta}{n\pi}}\right\}.$$

## § 5. Series of Complex Terms

The infinite series

$$\Sigma w_n \equiv w_1 + w_2 + w_3 + \ldots,$$

in which $\quad w_n = u_n + iv_n, \quad n = 1, 2, 3, \ldots,$

$u_n$ and $v_n$ being real, is said to be convergent if each of the series $\Sigma u_n$ and $\Sigma v_n$ is convergent. If the sums of these series are U and V respectively, the series $\Sigma w_n$ converges to the sum W, where $W = U + iV$.

*Condition for Convergence.*—The necessary and sufficient condition that the series $\Sigma w_n$ should be convergent is that, corresponding to any assigned positive number $\epsilon$, however small, a positive integer $m$ can be found such that, if $n \geqq m$,

$$| w_{n+1} + w_{n+2} + \ldots + w_{n+p} | < \epsilon,$$

where $p$ is any positive integer.

The condition is necessary; for, if $U_n$, $V_n$, $W_n$ are the sums of the first $n$ terms of the series $\Sigma u_n$, $\Sigma v_n$, $\Sigma w_n$ respectively,

$$W_n = U_n + iV_n,$$

and $\quad W_{n+p} - W_n = (U_{n+p} - U_n) + i(V_{n+p} - V_n).$

Hence

$$| W_{n+p} - W_n | \leqq | U_{n+p} - U_n | + | V_{n+p} - V_n |.$$

Now if the series $\Sigma u_n$ and $\Sigma v_n$ are convergent, $m$ can be chosen so large that each term on the right of this inequality is less than $\frac{1}{2}\epsilon$, and therefore $| W_{n+p} - W_n | < \epsilon$. Thus the condition is necessary.

Also the condition is sufficient. For

$$|U_{n+p} - U_n| \leqq |W_{n+p} - W_n|,$$

and
$$|V_{n+p} - V_n| \leqq |W_{n+p} - W_n|,$$

and consequently, if the condition is fulfilled, $\Sigma u_n$ and $\Sigma v_n$ are convergent.

*Absolute Convergence.*—The series $\Sigma w_n$ is said to be absolutely convergent if each of the series $\Sigma u_n$ and $\Sigma v_n$ is absolutely convergent. From the inequalities

$$|w_n| \leqq |u_n| + |v_n|, n = 1, 2, 3, \ldots,$$

it follows that the series $\Sigma |w_n|$ is then also convergent.

Conversely, if the series $\Sigma |w_n|$ is convergent, the series $\Sigma w_n$ is absolutely convergent. This follows from the inequalities

$$|u_n| \leqq |w_n|, \ |v_n| \leqq |w_n|, n = 1, 2, 3, \ldots.$$

*Power Series.*—If, when $n$ tends to infinity, $|a_n/a_{n+1}|$ tends to R, it follows, as in Chapter XVIII., § 4, that the power series $\Sigma a_n z^n$ converges absolutely if $|z| < R$. The length R is called the *radius of convergence* and the circle $|z| = R$ is the *circle of convergence.*

*Example* 1.—Show that, if $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots.$$

$\left[\frac{1}{1-z} = 1 + z + z^2 + \ldots + z^{n-1} + \frac{z^n}{1-z},\right.$ and, when $|z| < 1$, $|z^n| \to 0$ as $n \to \infty$.]

*Example* 2.—Show that, if $-1 < r < 1$,

(i) $$\frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} = 1 + r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \ldots,$$

(ii) $$\frac{r\sin\theta}{1 - 2r\cos\theta + r^2} = r\sin\theta + r^2\sin 2\theta + r^3\sin 3\theta + \ldots.$$

[In Example 1 put $z = r(\cos\theta + i\sin\theta)$ and equate the real and the imaginary parts.]

*Example* 3.—Show that the series

$$1 + \frac{z}{1\,!} + \frac{z^2}{2\,!} + \frac{z^3}{3\,!} + \ldots$$

converges absolutely for all values of $z$.

*Example* 4.—Show that the series $F(\alpha, \beta\,;\ \gamma\,;\ z)$ is absolutely convergent if $|\,z\,| < 1$.

*Multiplication of Series.*—The rule for the product of two absolutely convergent series of complex terms is identical with that stated in Chapter XVIII, § 6, and the proof is that given there under Case II.

*Example* 5.—Show that

$$\sum_{n=0}^{\infty} \frac{z_1^{\,n}}{n\,!} \sum_{n=0}^{\infty} \frac{z_2^{\,n}}{n\,!} = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n\,!}.$$

*Functions of a Complex Variable.*—If $u(x, y)$ and $v(x, y)$ are real functions of the independent real variables $x$ and $y$, the function $u(x, y) + i\,v(x, y)$ is said to be a function of the complex variable $z$, where $z = x + iy$. It is usual, however, to confine the discussion of functions of a complex variable to those functions which can be expressed explicitly in terms of the variable $z$ (see Ch. XVI, § 2). In what follows the functions considered can all be expressed as convergent power series in $z$, or as the quotients of two such series.

If, to each value of $z$ in a region of the $z$-plane, there corresponds one and only one value of the function $f(z)$, that function is said to be *uniform* or *single-valued* in the region. If more than one value corresponds, in general, to each value of $z$, the function is *many-valued* or *multiple-valued.* For instance, if $n$ is a positive integer greater than unity, the function $z^{1/n}$ has $n$ values for each value of $z$ except $z = 0$. These are given in Chapter XIV, § 7, in the form

$$r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right),$$

where $z = r\,(\cos\theta + i\sin\theta)$ and $k = 0, 1, 2, \ldots, n-1$.

If a straight line along the negative real axis from 0 to $-\infty$ is taken as a barrier or *cross-cut* which the variable $z$ is not allowed to cross, any one of these values can be regarded as a uniform function throughout the plane, and it is called a *branch* of the function. When $z$ passes round the origin, crossing over the cross-cut, the function changes from one of its branches to another. For this reason the origin is called a *branch point* of the function.

*Limit of a Function.*—A function $f(z)$, or, if $f(z)$ is many-valued, any one branch of this function, is said to tend to the limit L when $z$ tends to $z_1$, if, corresponding to any assigned positive number $\epsilon$, however small, a positive number $\delta$ can be found such that, if $|z - z_1| \leqq \delta$, $z \neq z_1$,

$$|f(z) - f(z_1)| < \epsilon.$$

We then write

$$\underset{z \to z_1}{\mathcal{L}} f(z) = \mathrm{L}\,; \quad \text{or} \quad f(z) \to \mathrm{L} \quad \text{when} \quad z \to z_1.$$

*Continuity.*—A uniform function $f(z)$, or, if the function is many-valued, a branch of the function, is continuous at a point $z_1$ if (i) $f(z_1)$ has a value, (ii) $f(z) \to f(z_1)$ when $z \to z_1$.

*Uniform Convergence.*—The definitions of uniform convergence of sequences and series of functions of a complex variable $z$ are identical with those given in Chapter XIX, § 1, for functions of a real variable, except that the closed intervals are replaced by *closed regions* of the $z$-plane; that is, by regions which include the points on their boundaries. Theorem I, on the continuity of the sum of a series, can then be proved in the same way as before, and the proof of Weierstrass's M-Test in § 2 also remains valid. Thus, as in Chapter XIX, § 3, Corollary I, it can be shown that the sum of a power series is continuous at all points within the circle of convergence.

## § 6. The Generalised Exponential Function

The exponential function may be defined for complex values of the argument by the equation

$$\exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \qquad (5)$$

The series converges absolutely for all values of $z$, and, consequently, the function is continuous for all values of $z$. It is clearly a single-valued function.

*Example* 1.—Show that $\exp(z_1)\ \exp(z_2) = \exp(z_1 + z_2)$. [See § 5, Example 5.]

From (5),

$$\exp(ix) = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right),$$

and therefore, if $x$ is real,

$$\exp(ix) = \cos x + i \sin x. \qquad (6)$$

It follows that, if $z = x + iy$, where $x$ and $y$ are real,

$$\exp(z) = \exp(x)\exp(iy), = e^x(\cos y + i \sin y). \qquad (7)$$

It is often convenient to write $e^z$ in place of $\exp(z)$. The function $e^z$ so defined is, of course, a uniform function of $z$.

From the equation

$$e^z = e^x(\cos y + i \sin y)$$

it is easy to deduce the index laws

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \quad e^0 = 1, \quad e^{-z} = 1/e^z.$$

*Example* 2.—If $C = e^{r\cos\theta}\cos(r\sin\theta)$, $S = e^{r\cos\theta}\sin(r\sin\theta)$, express $C + iS$ as a function of $z$; and expand C and S in ascending powers of $r$, giving in each case the general term.

Ans. $C + iS = \exp(z)$,

$$C = \sum_{n=0}^{\infty} \frac{r^n \cos n\theta}{n!}, \quad S = \sum_{n=1}^{\infty} \frac{r^n \sin n\theta}{n!}.$$

*Periodicity.*—If $m$ is an integer or zero,

$$e^{2im\pi} = \cos 2m\pi + i \sin 2m\pi = 1.$$

Thus $$e^{z+2im\pi} = e^z,$$

so that the exponential function has the period $2i\pi$.

*Zeros and Infinities.*—From (7) it is clear that

$$| e^z | = e^x,$$

and therefore $e^z$ can only have zero or infinite values when $e^x$ is zero or infinite. Thus $e^z$ is zero only when $x = -\infty$, and infinite only when $x = +\infty$.

## § 7. The Generalised Circular and Hyperbolic Functions

Equation (6) may be written

$$e^{ix} = \cos x + i \sin x,$$

and, if $x$ is replaced by $-x$, this becomes

$$e^{-ix} = \cos x - i \sin x.$$

From these two equations it follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \quad . \quad (8)$$

These equations should be compared with those of Chapter XVII, § 9, for the hyperbolic functions.

The circular and hyperbolic functions may now be defined for complex values of their arguments by the equations

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad . \quad (9)$$

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}, \quad . \quad (10)$$

with corresponding expressions for the other circular and hyperbolic functions. From these definitions the relations

$$\cos (iz) = \cosh z, \quad \sin (iz) = i \sinh z, \quad . \quad (11)$$

$$\cosh (iz) = \cos z, \quad \sinh (iz) = i \sin z, \quad . \quad (12)$$

are easily derived. From (9) and (10) and the expansion (5), the expansions

$$\cos z = 1 - \frac{z^2}{2\,!} + \frac{z^4}{4\,!} - \ldots, \qquad (13)$$

$$\sin z = z - \frac{z^3}{3\,!} + \frac{z^5}{5\,!} - \ldots, \qquad (14)$$

$$\cosh z = 1 + \frac{z^2}{2\,!} + \frac{z^4}{4\,!} + \ldots, \qquad (15)$$

$$\sinh z = z + \frac{z^3}{3\,!} + \frac{z^5}{5\,!} + \ldots, \qquad (16)$$

can be deduced. From (9) and (10) it follows that

$$e^{iz} = \cos z + i \sin z, \quad e^z = \cosh z + \sinh z. \qquad (17)$$

*Example* 1.—Show that

(i) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$,
(ii) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$,
(iii) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$,
(iv) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.

*Example* 2.—Show that

(i) $\cos(z + n\pi) = (-1)^n \cos z$,
(ii) $\sin(z + n\pi) = (-1)^n \sin z$.

*Example* 3.—Show that

(i) $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$,
(ii) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$,
(iii) $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$,
(iv) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$.

*Example* 4.—Show that

$$\mathop{\mathcal{L}}_{z \to 0} \frac{\sin az}{z} = \mathop{\mathcal{L}}_{z \to 0} \frac{\sinh az}{z} = a.$$

[Employ formulæ (14) and (16).]

*Example* 5.—If $|z| < 1$, show that

(i) $|\sin z| \leqq \frac{6}{5}|z|$, (ii) $|\cos z| < 2$.

$$\Big[\text{From (14),} \quad |\sin z| \leqq |z| + \frac{|z|^3}{3\,!} + \ldots$$

$$\leqq |z|\left(1 + \frac{1}{6} + \frac{1}{6^2} + \ldots\right) = \frac{6}{5}|z|$$

From (13), $|\cos z| \leqq 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots = 2.]$$

*Example* 6.—If $|z| < 1$, show that

$$\left|\frac{z}{\sin z}\right| < \frac{\pi}{\sqrt{2}}.$$

$$\begin{aligned}[|\sin z| = |\sin(x+iy)| &= \sqrt{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)} \\ &= \sqrt{(\sin^2 x + \sinh^2 y)} \\ &\geqq \sqrt{(\sin^2 x + \sin^2 y)}, \text{ from (14) and (16)} \\ &\geqq \sqrt{\{1 - \cos(|x| + |y|)\cos(|x| - |y|)\}}.\end{aligned}$$

Now

$$(|x| + |y|)^2 = 2(x^2 + y^2) - (|x| - |y|)^2 \leqq 2(x^2 + y^2) < 2,$$

and therefore $|x| + |y| < \sqrt{2} < \frac{1}{2}\pi$.

Hence $-\frac{1}{2}\pi < |x| - |y| < \frac{1}{2}\pi$

and therefore

$$\begin{aligned}|\sin z| &\geqq \sqrt{\{1 - \cos(|x| + |y|)\}} \\ &\geqq \sqrt{(1 - \cos|z|)} = \sqrt{2}\sin\tfrac{1}{2}|z|.\end{aligned}$$

Thus

$$\left|\frac{z}{\sin z}\right| \leqq \frac{|z|}{\sqrt{2}\,.\sin\frac{1}{2}|z|} = \sqrt{2}\,\frac{\frac{1}{2}|z|}{\sin\frac{1}{2}|z|} \leqq \frac{\pi}{\sqrt{2}}.$$

(See p. 425, footnote.)]

*Example* 7.—Show that

(i) $\cos^2 z + \sin^2 z = 1$, (ii) $\cosh^2 z - \sinh^2 z = 1$.

*Example* 8.—Show that $(\cos z + i\sin z)(\cos z - i\sin z) = 1$.

*Example* 9.—Prove that

(i) $(\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)\ldots(\cos z_n + i\sin z_n)$
$= \cos(z_1 + z_2 + \ldots + z_n)$
$+ i\sin(z_1 + z_2 + \ldots + z_n)$,

(ii) $(\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)\ldots(\cos z_n - i\sin z_n)$
$= \cos(z_1 + z_2 + \ldots + z_n)$
$- i\sin(z_1 + z_2 + \ldots + z_n)$.

*Example* 10.—**Demoivre's Theorem.** Show that, if $n$ is an integer, positive or negative,

$$(\cos z + i\sin z)^n = \cos nz + i\sin nz\,;$$

while, if $n$ is a rational fraction, $\cos nz + i\sin nz$ is one of the values of $(\cos z + i\sin z)^n$.

*Example* 11.—Show that, if $n$ is an integer, positive or negative,

$$(\cos z - i \sin z)^n = \cos nz - i \sin nz\,;$$

while, if $n$ is a rational fraction, $\cos nz - i \sin nz$ is one of the values of $(\cos z - i \sin z)^n$.

*Example* 12.—If $n$ is a positive integer, show that

$$\text{(i)}\ \cos nz = \cos^n z\,(1 - {}^nC_2 t^2 + {}^nC_4 t^4 - \ldots),$$
$$\text{(ii)}\ \sin nz = \cos^n z\,({}^nC_1 t - {}^nC_3 t^3 + \ldots),$$
$$\text{(iii)}\ \tan nz = \frac{{}^nC_1 t - {}^nC_3 t^3 + \ldots}{1 - {}^nC_2 t^2 + {}^nC_4 t^4 - \ldots},$$

where $t = \tan z$.

[Formulæ (i) and (ii) are obtained by expanding the expressions on the right of the equations

$$2 \cos nz = (\cos z + i \sin z)^n + (\cos z - i \sin z)^n,$$
$$2\, i \sin nz = (\cos z + i \sin z)^n - (\cos z - i \sin z)^n.]$$

*Example* 13.—Show that

$$\cos(z_1 + z_2 + \ldots + z_n) = \cos z_1 \cos z_2 \ldots \cos z_n\,(1 - T_2 + T_4 - \ldots),$$
$$\sin(z_1 + z_2 + \ldots + z_n) = \cos z_1 \cos z_2 \ldots \cos z_n\,(T_1 - T_3 + T_5 - \ldots),$$

where $T_r$ is the sum of the products of $\tan z_1, \tan z_2, \ldots, \tan z_n$, taken $r$ at a time.

*Example* 14.—Show that

$$2^9 \sin^4 z \cos^6 z = \cos 10z + 2 \cos 8z - 3 \cos 6z - 8 \cos 4z + 2 \cos 2z + 6.$$

[Put $2i \sin z = e^{iz} - e^{-iz}$, $2 \cos z = e^{iz} + e^{-iz}$.]

*Example* 15.—Show that the factor formulæ of Chapter XV, § 5, all hold when $\theta$ is replaced by a complex number $z$.

*Example* 16.—If $n$ is a positive integer, and $z \neq k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$, show that

$$\cot z = \frac{1}{2n+1} \cot \frac{z}{2n+1} + \frac{2}{2n+1} \sum_{r=1}^{n} \frac{\sin \dfrac{z}{2n+1} \cos \dfrac{z}{2n+1}}{\sin^2 \dfrac{z}{2n+1} - \sin^2 \dfrac{r\pi}{2n+1}}.$$

As in Example 12 it can be shown that

$$\cot(2n+1)z = \frac{1 - {}^{2n+1}C_2\, t^2 + \ldots + (-1)^n\, {}^{2n+1}C_{2n}\, t^{2n}}{{}^{2n+1}C_1\, t - {}^{2n+1}C_3\, t^3 + \ldots + (-1)^n\, {}^{2n+1}C_{2n+1}\, t^{2n+1}},$$

where $t$ denotes $\tan z$. The factors of the denominator are of the form

$$t - \tan\{r\pi/(2n+1)\},$$

where

$$r = 0, \pm 1, \pm 2, \ldots, \pm n.$$

Hence

$$\cot(2n+1)z = \sum_{r=-n}^{n} \frac{A_r}{\tan z - \tan\dfrac{r\pi}{2n+1}},$$

where

$$A_r = \mathop{\mathcal{L}}_{z \to \frac{r\pi}{2n+1}} \left\{\cot(2n+1)z\left(\tan z - \tan\frac{r\pi}{2n+1}\right)\right\},$$

or, if

$$\zeta = z - r\pi/(2n+1),$$

$$A_r = \sec^2\frac{r\pi}{2n+1} \mathop{\mathcal{L}}_{\zeta \to 0} \{\cot(2n+1)\zeta \sin\zeta\}$$

$$= \sec^2\frac{r\pi}{2n+1} \Big/ (2n+1),$$

by Example 4.

The result can now be obtained by replacing $z$ by $z/(2n+1)$ and combining the terms in pairs.

*Example* 17.—**Expansion of cot z in an infinite series of partial fractions.** Show that, if $z \neq k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$

$$\cot z = \frac{1}{z} + \sum_{r=1}^{\infty} \frac{2z}{z^2 - r^2\pi^2}.$$

This can be deduced from the formula of Example 16 by applying Tannery's Theorem, which, as can be seen by referring to the proof, holds for complex as well as real series. The formulæ of Examples 4, 5 and 6 are required in the proof. The formula may also be deduced from Example 2, § 10.

## § 8. The Generalised Logarithmic Function

If $z = \exp w$, where $w = u + iv$, then

$$z = r\cos\theta + i\,r\sin\theta = e^{u+iv} = e^u\cos v + i\,e^u\sin v,$$

where, of course, $\theta$ has an infinite number of values differing from each other by multiples of $2\pi$: therefore

$$r = e^u \quad \text{and} \quad \theta = v.$$

Hence $$u = \log r \quad \text{and} \quad v = \theta,$$

so that $$w = \log r + i\theta.$$

This function is the inverse of $\exp w$, and is denoted by $\operatorname{Log} z$; thus

$$\operatorname{Log} z = \log r + i\theta \quad . \quad . \quad . \quad (18)$$

The function is infinitely many-valued, since, for each value of $z$, $\theta$ has an infinite number of values, differing from each other by multiples of $2\pi$. That branch of the function for which $-\pi < \theta \leqq \pi$ is called the *principal value*, and is denoted by $\log z$. When $\theta = 0$, $\log z$ is the ordinary Naperian logarithm of a positive real number. If $z$ passes round the origin in the positive direction, the value of $\operatorname{Log} z$ increases by $2\pi i$; while if $z$ passes round the origin in the negative direction the value of the function decreases by $2\pi i$. The origin is therefore said to be a *branch point* of the function.

*Example* 1.—Show that

$$\text{(i)}\ \operatorname{Log} z_1 + \operatorname{Log} z_2 = \log(z_1 z_2) + 2m\pi i,$$
$$\text{(ii)}\ \operatorname{Log} z_1 - \operatorname{Log} z_2 = \log(z_1/z_2) + 2n\pi i,$$

where $m$ and $n$ are integers or zero.

If $w = e^z$, $|w| = e^x$ and $\operatorname{amp} w = y$, so that

$$\operatorname{Log} w = \log e^x + iy + 2n\pi i = x + iy + 2n\pi i.$$

Thus

$$\operatorname{Log} e^z = z + 2n\pi i, \text{ where } n \text{ is zero or an integer.} \quad (19)$$

*Zeros and Infinities.*—Since $\log r$ is infinite when $r$ is zero or infinite, Log $z$ has infinities at the origin and at infinity. Log $z$ is only zero when $\log r$ and $\theta$ are both zero, that is, $\log z = 0$ when $z = 1$.

*Expansion in Series of log* $(1 + z)$.—If $z = re^{i\theta}$,

$$\log (1 + z) = \log (1 + r\cos\theta + ir\sin\theta)$$
$$= \tfrac{1}{2}\log(1 + 2r\cos\theta + r^2) + i\tan^{-1}\left(\frac{r\sin\theta}{1 + r\cos\theta}\right).$$

If $0 < r < 1$, the inverse tangent lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, as its cosine has the sign of $(1 + r\cos\theta)$, which is then always positive.

Now, in Chapter XIX, § 3, Example 5, put $-r$ for $r$; thus, if $|r| < 1$,

$$\tfrac{1}{2}\log(1 + 2r\cos\theta + r^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{r^n}{n}\cos n\theta,$$

and

$$\tan^{-1}\left(\frac{r\sin\theta}{1 + r\cos\theta}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{r^n}{n}\sin n\theta.$$

Hence, if $|z| < 1$,

$$\log(1 + z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{r^n}{n}(\cos n\theta + i\sin n\theta),$$

or $$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots \qquad (20)$$

*Limits.*—From (20) it follows that

$$\mathop{\mathcal{L}}_{z\to 0}\left\{\frac{1}{z}\log(1 + z)\right\} = 1. \qquad (21)$$

Now let $\mu$ be a fixed number, and let $z$ and $\nu$ be variable quantities such that, when $z \to 0$, $\nu z \to \mu$ and $|\nu| \to \infty$; then, as in Chapter XVII, § 6, it follows that

$$\mathop{\mathcal{L}}_{z\to 0}\{\nu\log(1 + z)\} = \mu. \qquad (22)$$

*Abel's Theorem.*—If $S(z)$ is the sum of the series $\Sigma a_n z^n$ whose radius of convergence is R, and if the series converges at a point $z_0$ on the circle of convergence,

$$\mathop{\mathcal{L}}_{z \to z_0} S(z) = S(z_0),$$

where $z \to z_0$ along a radius. It is assumed that the coefficients $a_n$ are all real.

Let $z = r(\cos\theta + i\sin\theta)$, $z_0 = R(\cos\theta_0 + i\sin\theta_0)$;

then $$S(z) = U(r, \theta) + iV(r, \theta),$$

where $U(r, \theta) = \Sigma a_n \cos n\theta\, r^n$, $V(r, \theta) = \Sigma a_n \sin n\theta\, r^n$,

and, if $z$ lies on the radius through $z_0$,

$$U(r, \theta_0) = \Sigma a_n \cos n\theta_0\, r^n, \; V(r, \theta_0) = \Sigma a_n \sin n\theta_0\, r^n.$$

The series $U(r, \theta_0)$, $V(r, \theta_0)$, regarded as power series in $r$, are convergent for $-R < r \leq R$. Hence (Ch. XIX, § 4), when $r \to R$,

$$U(r, \theta_0) \to U(r_0, \theta_0), \quad V(r, \theta_0) \to V(r_0, \theta_0),$$

and, consequently,

$$U(r, \theta_0) + iV(r, \theta_0) \to U(r_0, \theta_0) + iV(r_0, \theta_0).$$

That is, when $z \to z_0$ along a radius,

$$S(z) \to S(z_0).$$

*Example* 2.—Show that the expansion

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

is valid at all points of the circle of convergence $|z| = 1$ except the point $-1$.

On the circle $z = e^{i\theta}$, and the series is equal to $U + iV$, where

$$U = \cos\theta - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \cdots = -\sum \frac{\cos n(\theta + \pi)}{n},$$

$$V = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \cdots = -\sum \frac{\sin n(\theta + \pi)}{n}.$$

Now (Ch. XVIII, § 5, Examples 1, 2) those series are convergent if $-\pi < \theta < \pi$. Thus the given series is convergent, and the result follows by Abel's Theorem.

*Example* 3.—Show that the expansion

$$\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$$

is valid at all points of the circle $|z| = 1$ for which $-\frac{1}{2}\pi < \operatorname{amp} z < \frac{1}{2}\pi$.

*Example* 4.—If $|z| \leqq 1$, $-\frac{1}{2}\pi < \operatorname{amp} z < \frac{1}{2}\pi$, mark the points $\mathrm{P}(z)$, $\mathrm{A}(-1)$, $\mathrm{Q}(iz)$, $\mathrm{R}(-iz)$ on a diagram, and show that

$$\operatorname{amp}\left(\frac{1+iz}{1-iz}\right) = \mathrm{R\hat{A}Q}.$$

Hence show that, if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,

(i) $\frac{\pi}{4} = \frac{\cos\theta}{1} - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \ldots,$

(ii) $\frac{1}{2}\log(\sec\theta + \tan\theta) = \frac{\sin\theta}{1} - \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} - \ldots.$

*Example* 5.—**The Inverse Tangent.** Show that

$$\tan^{-1} z = \frac{1}{2i}\operatorname{Log}\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) + m\pi,$$

where $m$ is an integer or zero.

[Let $z = \tan w = (e^{iw} - e^{-iw})/\{i(e^{iw} + e^{-iw})\}$.

Then $e^{2iw} = (1+iz)/(1-iz)$.]

## § 9. The Generalised Power

In Chapter XVII, § 5, the function $a^x$, where $a$ is real and positive and $x$ is real, was defined by the equation

$$a^x = e^{x \log a}.$$

The function so defined has only one value, which is real and positive. We are now in a position to give a definition of the function in which all restrictions have been removed.

If $a$ and $z$ are any numbers, real or complex, $a^z$ is defined by the equation

$$a^z = e^{z \operatorname{Log} a} = e^{z(\log a + 2m\pi i)}, \qquad (23)$$

where $m$ is an integer or zero.

If $z$ is an integer, the definition gives one value of the function only.

If $z = p/q$, where $p$ and $q$ are integers with no common factor, and $q$ is positive, the definition gives $q$ distinct values of the function. The reader should verify that this is in agreement with the results of Chapter XIV, § 7.

If $z$ is irrational or complex, the definition gives an infinite number of values of the function. That value for which $m = 0$ is called the *principal value.*

From (23) and (19) it follows that

$$\operatorname{Log} a^z = z \operatorname{Log} a = z(\log a + 2m\pi i), \qquad (24)$$

where $m$ is an integer or zero.

It is now possible to give Demoivre's Theorem in its most general form.

*Demoivre's Theorem.*—If $z$ and $\nu$ are any numbers, real or complex, and $m$ is an integer or zero,

$$\cos \nu(z + 2m\pi) + i \sin \nu(z + 2m\pi)$$

is a value of $(\cos z + i \sin z)^\nu$. If $\nu$ is irrational or complex this gives an infinite number of values of the function.

For

$$\begin{aligned}(\cos z + i \sin z)^\nu &= (e^{iz})^\nu, \text{ by (17)}\\ &= e^{\nu \operatorname{Log}\{\exp(iz)\}}, \text{ by (23)}\\ &= e^{\nu i(z + 2m\pi)}, \text{ by (19)},\end{aligned}$$

from which the theorem follows. If $\nu$ is integral or rational this agrees with the theorem, as stated in Chapter XIV, § 6. If $\nu$ is irrational or complex $(\cos z + i \sin z)^\nu$ has an infinite number of values.

Corollary 1.—$\cos \nu(z + 2m\pi) - i \sin \nu(z + 2m\pi)$ is one value of $(\cos z - i \sin z)^\nu$.

COROLLARY 2.—If $z = r(\cos\theta + i\sin\theta)$, and if $n$ is any number, real or complex,

$$z^n = e^{n\log r}\{\cos n(\theta + 2m\pi) + i\sin n(\theta + 2m\pi)\},$$

where $m$ is integral or zero.

*Example* 1.—Show that $e^{z_1 z_2}$ is one value of $(e^{z_1})^{z_2}$.

*The Exponential Limit.*—If $z$ and $\nu$ are variable numbers, real or complex, and if $\mu$ is a fixed number, real or complex, such that, when $z \to 0$, $z\nu \to \mu$, and $|\nu| \to \infty$, and if $(1+z)^\nu$ has its principal value,

$$\mathop{\mathcal{L}}_{z\to 0}(1+z)^\nu = e^\mu \quad . \quad . \quad . \quad (25)$$

For $(1+z)^\nu = e^{\nu\log(1+z)}$, and, by (22), when $z \to 0$, $\nu\log(1+z) \to \mu$.

*Example* 2.—Show that, if $z \neq ik\pi$, where

$$k = 0, \pm 1, \pm 2, \ldots,$$

$$\coth z = \frac{1}{z} + \sum_{r=1}^{\infty}\frac{2z}{z^2 + r^2\pi^2}.$$

[The proof of Example 6, § 2, applies here also.]

## § 10. The Hypergeometric Function

By applying the test ratio it can be seen that the hypergeometric series $F(\alpha, \beta; \gamma; z)$ is absolutely convergent for $|z| < 1$, even if $\alpha$, $\beta$, $\gamma$ and $z$ are complex. When $|z| = 1$ the convergence can be investigated by means of the following extension of the theorem of Chapter XVIII, § 4.

THEOREM.—If

$$u_n = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{(\beta+1)(\beta+2)\ldots(\beta+n)},$$

where $\beta$ is not a negative integer then, for all values of $n$,

$$| u_n | < \frac{A}{n^{R(\beta - \alpha)}},$$

A being a (positive) constant independent of $n$, and $R(\beta - \alpha)$ denoting the real part of $\beta - \alpha$.

The proofs of the three lemmas of Chapter XVIII, § 4, hold when $\alpha$ and $\beta$ are complex; the theorem can be deduced, as was done there, from Lemma III.

On applying the theorem to the hypergeometric series, we find that, if $R(\gamma - \alpha - \beta) > 0$, it converges absolutely when $| z | = 1$; while, if $\alpha$, $\beta$ and $\gamma$ are real, and

$$-1 < \gamma - \alpha - \beta \leqq 0,$$

it converges conditionally for $| z | = 1$, provided that $z \neq 1$.

*Example* 1.—THE BINOMIAL THEOREM. If $| z | < 1$, and if $(1 + z)^m$, where $z$ and $m$ are real or complex, has its principal value, show that

$$(1 + z)^m = 1 + \frac{m}{1\,!} z + \frac{m(m-1)}{2\,!} z^2 + \frac{m(m-1)(m-2)}{3\,!} z^3 + \dots .$$

Investigate the validity of the expansion when $| z | = 1$.

[The proof of the theorem of Chapter XVIII, § 7, holds for complex series; and, consequently, the proof of the binomial theorem given in the example in that section holds when $z$ and $m$ are complex. From the theorem given above it can be seen that, when $| z | = 1$, the series converges absolutely if $R(m) > 0$; while, if $m$ is real and $-1 < m \leqq 0$, it converges conditionally for $| z | = 1$ unless when $z = -1$.]

*Example* 2.—Show that the following expansions

(i) $$\{2 \cos (\tfrac{1}{2}\theta - k\pi)\}^m \cos (\tfrac{1}{2}m\theta - mk\pi) = 1 + \frac{m}{1\,!} \cos \theta + \frac{m(m-1)}{2\,!} \cos 2\theta + \dots ,$$

(ii) $$\{2 \cos (\tfrac{1}{2}\theta - k\pi)\}^m \sin (\tfrac{1}{2}m\theta - mk\pi) = \frac{m}{1\,!} \sin \theta + \frac{m(m-1)}{2\,!} \sin 2\theta + \dots ,$$

(iii) $\{2\cos(\frac{1}{2}\theta - k\pi)\}^m \cos(\alpha - \frac{1}{2}m\theta + mk\pi)$

$$= \cos\alpha + \frac{m}{1!}\cos(\alpha - \theta) + \frac{m(m-1)}{2!}\cos(\alpha - 2\theta) + \ldots,$$

where $k$ is an integer, hold for $(2k-1)\pi \leqq \theta \leqq (2k+1)\pi$ if $R(m) > 0$, and for $(2k-1)\pi < \theta < (2k+1)\pi$ if $m$ is real and $-1 < m \leqq 0$.

*Example* 3.—If $m$ is real and greater than $-1$, show that

(i) $\{2\cos(\theta - k\pi)\}^m \cos(\alpha - m\theta + mk\pi)$

$$= \cos\alpha + \frac{m}{1!}\cos(\alpha - 2\theta) + \frac{m(m-1)}{2!}\cos(\alpha - 4\theta) + \ldots,$$

where $k$ is an integer, and $(k - \frac{1}{2})\pi < \theta < (k + \frac{1}{2})\pi$,

(ii) $$(2\cos\theta)^m \cos(2ms\pi) = \cos m\theta + \frac{m}{1!}\cos(m-2)\theta + \frac{m(m-1)}{2!}\cos(m-4)\theta + \ldots,$$

where $s$ is an integer and $(2s - \frac{1}{2})\pi < \theta < (2s + \frac{1}{2})\pi$,

(iii) $$(-2\cos\theta)^m \cos\{(2s+1)m\pi\} = \cos m\theta + \frac{m}{1!}\cos(m-2)\theta + \frac{m(m-1)}{2!}\cos(m-4)\theta + \ldots,$$

where $(2s + \frac{1}{2})\pi < \theta < (2s + \frac{3}{2})\pi$,

(iv) $$(2\cos\theta)^m \sin(2ms\pi) = \sin m\theta + \frac{m}{1!}\sin(m-2)\theta + \frac{m(m-1)}{2!}\sin(m-4)\theta + \ldots,$$

where $(2s - \frac{1}{2})\pi < \theta < (2s + \frac{1}{2})\pi$,

(v) $$(-2\cos\theta)^m \sin\{(2s+1)m\pi\} = \sin m\theta + \frac{m}{1!}\sin(m-2)\theta + \frac{m(m-1)}{2!}\sin(m-4)\theta + \ldots,$$

where $(2s + \frac{1}{2})\pi < \theta < (2s + \frac{3}{2})\pi$.

*Example* 4.—If $m$ is real and $> -1$, show that,

(i) $(2\sin\theta)^m \cos\{(2s + \frac{1}{2})m\pi\}$

$$= \cos m\theta - \frac{m}{1!}\cos(m-2)\theta + \frac{m(m-1)}{2!}\cos(m-4)\theta - \ldots,$$

where $2s\pi < \theta < (2s+1)\pi$,

(ii) $(-2\sin\theta)^m \cos\{(2s+\tfrac{3}{2})m\pi\}$

$$= \cos m\theta - \frac{m}{1\,!}\cos(m-2)\theta + \frac{m(m-1)}{2\,!}\cos(m-4)\theta - \ldots,$$

where $(2s+1)\pi < \theta < (2s+2)\pi$,

(iii) $(2\sin\theta)^m \sin\{(2s+\tfrac{1}{2})m\pi\}$

$$= \sin m\theta - \frac{m}{1\,!}\sin(m-2)\theta + \frac{m(m-1)}{2\,!}\sin(m-4)\theta - \ldots,$$

where $2s\pi < \theta < (2s+1)\pi$,

(iv) $(-2\sin\theta)^m \sin\{(2s+\tfrac{3}{2})m\pi\}$

$$= \sin m\theta - \frac{m}{1\,!}\sin(m-2)\theta + \frac{m(m-1)}{2\,!}\sin(m-4)\theta - \ldots,$$

where $(2s+1)\pi < \theta < (2s+2)\pi$.

## § 11. Infinite Products

As the exponential function is continuous for complex values of the argument, it follows, as in the theorem of § 3, that the infinite product $\Pi f_r$ is convergent if, and only if, the series $\Sigma \log f_r$ is convergent.

*Infinite Product for sin z.*—If $m$ is a positive integer, such that $|z| < m\pi$, and if $r \geq m$,

$$1 - \frac{z^2}{r^2\pi^2} = e^{\log\left(1 - \frac{z^2}{r^2\pi^2}\right)}$$

It follows that the product

$$z\prod_{r=1}^{\infty}\left(1 - \frac{z^2}{r^2\pi^2}\right)$$

is convergent if the series $\displaystyle\sum_{r=m}^{\infty} \log\left(1 - \frac{z^2}{r^2\pi^2}\right)$ is convergent.

The convergence of the series can be established as in § 3, Example 9. Since $m$ can always be chosen so that $|z| < m\pi$, the product converges for all values of $z$.

Now, if $|z| < m\pi$, each of the series

$$\log\left(1 - \frac{z^2}{r^2\pi^2}\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z^2}{r^2\pi^2}\right)^n,\ r = m, m+1, m+2, \ldots$$

is absolutely convergent, and the series

$$\sum_{r=m}^{\infty}\left\{-\log\left(1-\frac{|z|^2}{r^2\pi^2}\right)\right\}$$

is also convergent. Hence, by the theorem of Chapter XVIII, § 7, the series

$$\sum_{r=m}^{\infty}\log\left(1-\frac{z^2}{r^2\pi^2}\right)$$

can be rearranged in powers of $z$, giving a series $\sum_{n=1}^{\infty} B_n z^{2n}$ which is absolutely convergent for $|z| < m\pi$. By a further application of the theorem, since the exponential series converges for all values of its argument, it follows that the product

$$\prod_{r=m}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right)=e^{\sum_{n=1}^{\infty} B_n z^{2n}}$$

can be expressed as a series $\sum_{n=0}^{\infty} C_n z^{2n}$ which converges absolutely for $|z| < m\pi$. Thus, on multiplying by the remaining factors, we find that the given product can be expressed as a series in the form

$$z\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right)=\sum_{n=0}^{\infty} D_n z^{2n+1},$$

the series converging absolutely for $|z| < m\pi$.

Now, when $z$ is real, we know that the product is equal to $\sin z$. Also, when $z$ is real, $\sin z$ can be expanded uniquely [Ch. XVI, (3)] in the series

$$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots,$$

which converges for all values of $z$. Hence

$$D_n = (-1)^n/(2n+1)!,$$

and, consequently, if $|z| < m\pi$,

$$z\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots,$$

or, by formula (14),

$$\sin z = z\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2\pi^2}\right). \qquad (26)$$

Now, no matter what value $z$ may have, an $m$ can be found such that $|z| < m\pi$. Thus formula (26) holds for all values of $z$.

*Alternative Proof.*—Formula (26) may also be established by means of Tannery's Theorem for products (§ 3), the proof of which is still valid when the factors of the product are complex. The theorem is applied to the formula

$$\sin z = n\sin\frac{z}{n}\prod_{r=1}^{n-1}\left(1-\sin^2\frac{z}{2n}\Big/\sin^2\frac{r\pi}{2n}\right).$$

If $n$ is taken so large that $2n > |z|$, it follows from § 7, Example 5, that

$$\left|\sin\frac{z}{2n}\right| \leqq \frac{6}{5}\frac{|z|}{2n}.$$

Hence, with the aid of the inequality (see § 2)

$$\sin\frac{r\pi}{2n} > \frac{r}{n},$$

we find that

$$\left|\sin^2\frac{z}{2n}\Big/\sin^2\frac{r\pi}{2n}\right| < M_r,$$

where
$$M_r = \frac{9|z|^2}{25r^2}.$$

But the series $\Sigma M_r$ is convergent: hence Tannery's Theorem is applicable.

## § 12. Applications of Dirichlet's Integrals

Dirichlet's Integrals * are of fundamental importance in the theory of Fourier Series. They may also be employed to evaluate various trigonometric series.

*Definition.*—A function $f(x)$ is said to satisfy *Dirichlet's Conditions* in a given interval if

(i) the function is continuous at all points of the interval, except possibly at a finite number of points at which it possesses finite discontinuities, and

(ii) the function has only a finite number of turning points in the interval.

*Properties of the Integrals.*—If $f(x)$ satisfies Dirichlet's Conditions in the intervals of integration,

$$\mathop{\mathcal{L}}_{m \to \infty} \int_a^b f(x) \sin mx \, dx = 0, \quad . \quad . \quad . \quad (27)$$

$$\mathop{\mathcal{L}}_{m \to \infty} \int_a^b f(x) \cos mx \, dx = 0, \quad . \quad . \quad . \quad (28)$$

and

$$\mathop{\mathcal{L}}_{m \to \infty} \int_0^a f(x) \frac{\sin mx}{x} \, dx = \tfrac{1}{2}\pi f(0+), \quad . \quad (29)$$

where, in (29), $0 < a$ and $f(0+)$ is the limit of $f(x)$ when $x \to 0$ through positive values.

The integrals in (27), (28) and (29) are Dirichlet's Integrals. These formulæ are valid even when $f(x)$ is a complex function of $x$, as they hold for the real and imaginary parts of $f(x)$ separately.

* Discussions of the properties of Dirichlet's Integrals are to be found in many text-books; among others, Gibson, *Elementary Treatise on the Calculus*, Chapter XXII, and MacRobert, *Spherical Harmonics*, Chapter I.

*Applications to Trigonometric Series.*—The identity

$$\sum_{n=-m}^{m} \cos (n+\alpha)\theta = \cos (\alpha\theta) \frac{\sin (m+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta},$$

where $m$ is a positive integer and $\alpha$ is any number, real or complex, leads, on integration, to the equation

$$\sum_{n=-m}^{m} \frac{\sin (n+\alpha)\theta}{n+\alpha} = \int_0^\theta \cos (\alpha\theta) \frac{\theta}{\sin \frac{1}{2}\theta} \frac{\sin (m+\frac{1}{2})\theta}{\theta} d\theta.$$

If $0 < \theta < 2\pi$, the integral on the right is of the form (29), and gives, when $m \to \infty$,

$$\sum_{n=-\infty}^{\infty} \frac{\sin (n+\alpha)\theta}{n+\alpha} = \pi. \qquad . \qquad . \qquad (30)$$

*Example* 1.—Show that

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{\pi-\theta}{2}, \quad 0 < \theta < 2\pi.$$

[Make $\alpha \to 0$ in (30).]

Again, the identity

$$\sum_{n=-m}^{m} \sin (n+\alpha)\theta = \sin (\alpha\theta) \frac{\sin (m+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta}$$

leads to the equation

$$\sum_{n=-m}^{m} \frac{1-\cos (n+\alpha)\theta}{n+\alpha} = \int_0^\theta \sin (\alpha\theta) \frac{\theta}{\sin \frac{1}{2}\theta} \frac{\sin (m+\frac{1}{2})\theta}{\theta} d\theta.$$

If the L.H.S. is written in the form

$$\left\{\frac{1}{\alpha} + \sum_{n=1}^{m} \frac{2\alpha}{\alpha^2 - n^2}\right\} - \sum_{n=-m}^{m} \frac{\cos (n+\alpha)\theta}{n+\alpha},$$

where $\alpha$ is not an integer, it follows from § 7, Example 17 and (29) that

$$\sum_{n=-\infty}^{\infty} \frac{\cos (n+\alpha)\theta}{n+\alpha} = \pi \cot (\alpha\pi), \quad 0 < |\theta| < 2\pi. \quad (31)$$

*Example* 2.—Show that, if $\alpha$ is not an integer and $0<\theta<2\pi$,

$$\text{(i)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{i(n+\alpha)\theta} = \frac{\pi}{\sin(\alpha\pi)} e^{i\alpha\pi},$$

$$\text{(ii)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{i(n+\beta)\theta} = \frac{\pi}{\sin(\alpha\pi)} e^{i\alpha\pi+i(\beta-\alpha)\theta},$$

$$\text{(iii)} \quad \sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\theta}{n+\alpha} = \frac{\pi}{\sin(\alpha\pi)} \cos\{\alpha\pi+(\beta-\alpha)\theta\},$$

$$\text{(iv)} \quad \sum_{n=-\infty}^{\infty} \frac{\sin(n+\beta)\theta}{n+\alpha} = \frac{\pi}{\sin(\alpha\pi)} \sin\{\alpha\pi+(\beta-\alpha)\theta\}.$$

*Example* 3.—Show that, if $x \neq 0$, $0<\theta<2\pi$,

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{n \sin n\theta}{n^2+x^2} = \frac{\pi \sinh(\pi-\theta)x}{2\sinh(\pi x)},$$

$$\text{(ii)} \quad \frac{1}{2x} + \sum_{n=1}^{\infty} \frac{x\cos n\theta}{n^2+x^2} = \frac{\pi\cosh(\pi-\theta)x}{2\sinh(\pi x)}.$$

[In Example 2, (ii), put $\beta = 0$, $\alpha = \pm ix$, adding and subtracting the equations so obtained.]

*Example* 4.—If $\alpha$ is not an integer, and $0<\theta<2\pi$, show that

$$\text{(i)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} e^{i(n+\alpha)\theta} = \frac{\pi^2}{\sin^2(\alpha\pi)} + \frac{i\pi\theta}{\sin(\alpha\pi)} e^{i\alpha\pi},$$

$$\text{(ii)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} e^{i(n+\beta)\theta}$$

$$= \frac{\pi}{\sin(\alpha\pi)} \{\pi\cot(\alpha\pi) - i(\pi-\theta)\} e^{i\alpha\pi+i(\beta-\alpha)\theta},$$

$$\text{(iii)} \quad \sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\theta}{(n+\alpha)^2}$$

$$= \frac{\pi}{\sin(\alpha\pi)} \begin{bmatrix} \pi\cot(\alpha\pi)\cos\{\alpha\pi+(\beta-\alpha)\theta\} \\ +(\pi-\theta)\sin\{\alpha\pi+(\beta-\alpha)\theta\} \end{bmatrix},$$

(iv) $$\sum_{n=-\infty}^{\infty} \frac{\sin (n+\beta)\theta}{(n+\alpha)^2} = \frac{\pi}{\sin(\alpha\pi)} \begin{bmatrix} \pi \cot(\alpha\pi) \sin\{\alpha\pi + (\beta-\alpha)\theta\} \\ -(\pi-\theta)\cos\{\alpha\pi + (\beta-\alpha)\theta\} \end{bmatrix}.$$

*Example* 5.—If $0 \leqq \theta \leqq 2\pi$, show that

(i) $$\sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)\sin(\alpha\pi)\sin(\beta\pi)} \times \{e^{i\beta(\pi-\theta)}\sin(\alpha\pi) - e^{i\alpha(\pi-\theta)}\sin(\beta\pi)\},$$

(ii) $$\sum_{n=-\infty}^{\infty} \frac{\cos n\theta}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)\sin(\alpha\pi)\sin(\beta\pi)} \times \{\cos(\pi-\theta)\beta \,.\, \sin(\alpha\pi) - \cos(\pi-\theta)\alpha \,.\, \sin(\beta\pi)\},$$

(iii) $$\sum_{n=-\infty}^{\infty} \frac{\sin n\theta}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)\sin(\alpha\pi)\sin(\beta\pi)} \times \{\sin(\pi-\theta)\beta \,.\, \sin(\alpha\pi) - \sin(\pi-\theta)\alpha \,.\, \sin(\beta\pi)\},$$

(iv) $$\sum_{n=-\infty}^{\infty} \frac{y\cos n\theta}{(n+x)^2+y^2} = \frac{\pi}{\cosh(2\pi y) - \cos(2\pi x)} \times \{\cos(\theta x)\sinh(2\pi-\theta)y + \cos(2\pi-\theta)x \,.\, \sinh(\theta y)\},$$

(v) $$\sum_{n=-\infty}^{\infty} \frac{y\sin n\theta}{(n+x)^2+y^2} = \frac{\pi}{\cosh(2\pi y) - \cos(2\pi x)} \times \{\sin(2\pi-\theta)x \,.\, \sinh(\theta y) - \sin(\theta x)\sinh(2\pi-\theta)y\}.$$

*Example* 6.—Show that, if $0 < \theta < 2\pi$,

(i) $$\sum_{n=0}^{\infty} \frac{\sin(x+n)\theta}{x+n} = \frac{1}{2}\int_{\theta}^{\pi} \frac{\sin(x-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta + \sin(\pi x)\sum_{n=0}^{\infty}\frac{(-1)^n}{x+n},$$

(ii) $\sum_{n=0}^{\infty} \frac{\cos(x+n)\theta}{x+n}$

$$= \frac{1}{2}\int_0^{\pi} \frac{\cos(x-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta + \cos(\pi x)\sum_{n=0}^{\infty}\frac{(-1)^n}{x+n}.$$

[Hardy.]

## EXAMPLES XX

**1.** Show that, if $t > 0$,

$$\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} = \sum_{n=1}^{\infty} \frac{2t}{t^2+4n^2\pi^2} < \frac{t}{12}.$$

Deduce that

$$\int_0^{\infty}\left(\frac{1}{1-e^{-t}} - \frac{1}{2} - \frac{1}{t}\right)\frac{e^{-t}\,dt}{t} = 2\int_0^{\infty}\frac{\tan^{-1} t\,dt}{e^{2\pi t}-1}.$$

**2.** If $f(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots,$

show that, if $0 < x < 1$,

(i) $f(x) + f(1-x) = \frac{1}{6}\pi^2 - \log x \,.\, \log(1-x),$

(ii) $f(-x) + f\left(\frac{x}{1+x}\right) = -\frac{1}{2}\{\log(1+x)\}^2.$

**3.** Show that, if $p$ is not integral,

$$\frac{\pi}{\sin p\pi} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+p} + \frac{1}{n+1-p}\right).$$

**4.** Prove that

$$\sum_{n=-\infty}^{\infty}\frac{1}{(x+n\alpha)^2+y^2} = \frac{\pi}{\alpha y}\,\frac{\sinh\dfrac{2\pi y}{\alpha}}{\cosh\dfrac{2\pi y}{\alpha} - \cos\dfrac{2\pi x}{\alpha}}.$$

**5.** Show that

$$\sum_{n=1}^{\infty}\frac{2a}{(2n-1)^2+a^2} = \frac{\pi}{2}\tanh\frac{\pi a}{2}.$$

6. If $n$ is an odd positive integer, show that

$$\frac{1}{2}\left\{\left(1+\frac{x}{n}\right)^n-\left(1-\frac{x}{n}\right)^n\right\}=x\prod_{k=1}^{m}\left\{1+\frac{x^2}{n^2\tan^2\dfrac{k\pi}{n}}\right\},$$

where $m=\frac{1}{2}(n-1)$, and deduce that

$$\sinh x = x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right).$$

$$\left[\log\frac{1}{2}\left\{\left(1+\frac{x}{n}\right)^n-\left(1-\frac{x}{n}\right)^n\right\}=\log x+\sum_{k=1}^{m}u_k,\right.$$

where

$$u_k=\log\left\{1+\frac{x^2\cos^2 k\pi}{k^2\pi^2\left(\dfrac{\sin\dfrac{k\pi}{n}}{\dfrac{k\pi}{n}}\right)^2}\right\}<\log\left\{1+\frac{x^2}{4k^2}\right\}.$$

The result is now obtained by applying Tannery's Theorem.]

7. Prove that

$$\sum_{n=1}^{\infty}\frac{2}{n(n+1)}=\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right)=2.$$

8. Discuss the products

(i) $(1-\sin\theta)\left(1-2\sin\dfrac{\theta}{4}\right)\left(1-3\sin\dfrac{\theta}{9}\right)\ldots,$

(ii) $\dfrac{(\sin\theta+1)(\sin\theta+2)(\sin\theta+3)\ldots}{(\cos\theta+1)(\cos\theta+2)(\cos\theta+3)\ldots},$

for values of $\theta$ such that $-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$.

Ans. (i) diverges if $\theta<0$, converges to 1 if $\theta=0$, diverges to zero if $\theta>0$; (ii) diverges if $\theta>\frac{1}{4}\pi$, converges to 1 if $\theta=\frac{1}{4}\pi$, diverges to zero if $\theta<\frac{1}{4}\pi$.

9. Show that

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96},$ (ii) $\displaystyle\sum_{n=1}^{\infty}\frac{n(n-1)}{(2n-1)^4}=\frac{\pi^2}{32}\left(1-\frac{\pi^2}{12}\right).$

10. Prove that

$$\frac{1}{1^4} + \frac{3}{3^4} + \frac{7}{5^4} + \frac{13}{7^4} + \ldots = \frac{1}{32}\pi^3(1 + \tfrac{1}{4}\pi^2).$$

11. Show that

$$\sin \pi x = \pi x \prod_{n=-\infty}^{\infty}{}' \left\{\left(1 - \frac{x}{n}\right)e^{\frac{x}{n}}\right\}.$$

12. Prove that

$$\cos x - \cos y = 2\sin^2(\tfrac{1}{2}y)\left(1 - \frac{x^2}{y^2}\right)$$
$$\times \prod_{r=1}^{\infty}\left[\left\{1 - \frac{x^2}{(2r\pi + y)^2}\right\}\left\{1 - \frac{x^2}{(2r\pi - y)^2}\right\}\right].$$

13. Show that

$$\prod_{n=1}^{\infty}\left\{\left(1 - \frac{x}{n}\right)\left(1 + \frac{x}{n+1}\right)\right\} = \frac{\sin \pi x}{\pi x(x+1)}.$$

14. Show that

$$1 + \sin x = \tfrac{1}{8}(\pi + 2x)^2\left\{1 - \frac{(\pi + 2x)^2}{4^2\pi^2}\right\}^2\left\{1 - \frac{(\pi + 2x)^2}{8^2\pi^2}\right\}^2 \ldots.$$

15. Prove that

$$\frac{\sin(x - y)}{\sin x} = \left(1 - \frac{y}{x}\right)\prod_{n=1}^{\infty}\left\{\left(1 + \frac{y}{n\pi - x}\right)\left(1 - \frac{y}{n\pi + x}\right)\right\}.$$

16. Prove that, if $a$ is not an integer,

$$\prod_{n=-\infty}^{\infty}\left\{1 - \frac{x^2}{(n-a)^2}\right\} = \frac{\cos 2\pi x - \cos 2\pi a}{1 - \cos 2\pi a}.$$

17. Prove that

$$\cos(\tfrac{1}{2}\pi \sin\theta) = \tfrac{1}{4}\pi\cos^2\theta\left(1 + \frac{\cos^2\theta}{2 \cdot 4}\right)\left(1 + \frac{\cos^2\theta}{4 \cdot 6}\right) \ldots.$$

18. Show that

$$(1 - x)(1 + \tfrac{1}{3}x)(1 - \tfrac{1}{5}x)(1 + \tfrac{1}{7}x) \ldots = \sqrt{2} \cdot \sin \tfrac{1}{4}\pi(1 - x).$$

19. Show that

(i) $$\cos\frac{\pi x}{6} + \sqrt{3}\sin\frac{\pi x}{6} = (1+x)\left(1-\frac{x}{5}\right)\left(1+\frac{x}{7}\right)\left(1-\frac{x}{11}\right)\left(1+\frac{x}{13}\right) \times \left(1-\frac{x}{17}\right)\left(1+\frac{x}{19}\right)\ldots,$$

(ii) $$\cos\frac{\pi x}{3} + \frac{1}{\sqrt{3}}\sin\frac{\pi x}{3} = (1+x)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{4}\right)\left(1-\frac{x}{5}\right)\left(1+\frac{x}{7}\right)\left(1-\frac{x}{8}\right)\ldots$$

20. Show that

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \to \log 2,$$

as $n \to \infty$, and hence show that

$$\mathop{\mathcal{L}}_{n\to\infty}\left\{\left(1-\frac{x}{n+1}\right)\left(1-\frac{x}{n+2}\right)\ldots\left(1-\frac{x}{2n}\right)\right\} = 2^{-x}.$$

21. From the identity

$$\frac{(z-b)(z-c)}{(a-b)(a-c)} + \frac{(z-c)(z-a)}{(b-c)(b-a)} + \frac{(z-a)(z-b)}{(c-a)(c-b)} = 1,$$

deduce the identities

(i) $$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\cos 2(\theta-\alpha) + \frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\cos 2(\theta-\beta) + \frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\cos 2(\theta-\gamma) = 1,$$

(ii) $$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha) + \frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\sin 2(\theta-\beta) + \frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\sin 2(\theta-\gamma) = 0.$$

[Put $z = e^{2i\theta}$, $a = e^{2i\alpha}$, $b = e^{2i\beta}$, $\quad = e^{2i\gamma}$.]

22. Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{z^n + 1}$$

converges absolutely if $|z| > 1$.

23. Show that the radius of convergence of the series

$$1 + 1!\,z + 2!\,z^2 + 3!\,z^3 + \ldots$$

is zero.

24. Show that, of the two series

$$\sum_{n=-\infty}^{\infty}{}' \log\left(1 + \frac{z}{n}\right), \quad \sum_{n=-\infty}^{\infty}{}' \log\left\{\left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}}\right\},$$

where $z$ is not a real integer, the former is divergent, and the latter absolutely convergent.

25. Prove that, if $\theta \neq k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$,

(i) $\cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \ldots = 0$,

(ii) $\cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \cos^3\theta\sin 3\theta + \ldots = \cot\theta$.

26. Prove that the infinite series

$$\cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{2^2} + \frac{\cos 4x}{2^3} + \ldots$$

converges for all real values of $x$, and find its sum. Deduce by integration the sum of the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n \,.\, 2^{n-1}}.$$

Ans. $(4\cos x - 2)/(5 - 4\cos x)$; $2\tan^{-1}(3\tan\frac{1}{2}x) - x$.

27. If O is the origin of co-ordinates and OP is the vector

$$ae^{i\theta} + be^{i(\theta+\alpha)},$$

where $a$ and $b$ are real, show that when $\theta$ varies the locus of P is a circle with the origin as centre and radius of length

$$\sqrt{(a^2 + 2ab\cos\alpha + b^2)}.$$

28. Express $(x - ye^{i\theta})/(x + ye^{i\theta})$ in the form $a + ib$, where $a$ and $b$ are real.

Ans. $(x^2 - y^2 - i2xy\sin\theta)/(x^2 + 2xy\cos\theta + y^2)$.

29. Solve completely the equation

$$x^n + a^n e^{in\theta} = 0,$$

where $n$ is a positive integer.

Ans. $$x = a \exp i\left(\theta + \frac{2k+1}{n}\pi\right),$$

where $$k = 0, 1, 2, \ldots, n-1.$$

30. If $u = e^x(x \cos y - y \sin y)$, $v = e^x(x \sin y + y \cos y)$, express $u + iv$ as a function of $z$.

Ans. $ze^z$.

31. If $$u = e^{x-y} \cos (x+y) + e^{x+y} \cos (x-y),$$

and $$v = e^{x-y} \sin (x+y) - e^{x+y} \sin (x-y),$$

show that $$u + iv = 2e^z \cos z.$$

32. If $u = e^x (\sin x \cos y \cosh y - \cos x \sin y \sinh y)$,

and $$v = e^x(\sin x \sin y \cosh y + \cos x \cos y \sinh y),$$

show that $$u + iv = e^z \sin z.$$

33. If $x + iy = \sin (u + iv)$, show that the curves

$$u = \text{constant}, v = \text{constant}$$

are confocal central conics.

34. Show that

$$\frac{2(1-x)\cos\theta}{1 - 2x\cos 2\theta + x^2} = \frac{e^{i\theta}}{1 - xe^{2i\theta}} + \frac{e^{-i\theta}}{1 - xe^{-2i\theta}};$$

and deduce that, if $|x| < 1$,

$$\frac{(1-x)\cos\theta}{1 - 2x\cos 2\theta + x^2} = \cos\theta + x\cos 3\theta + x^2\cos 5\theta + \ldots.$$

35. $A_1, A_2, \ldots, A_n$ are the vertices of a regular polygon of $n$ sides inscribed in a circle whose centre is O and radius $a$. P is a point inside the circle such that OP $= d$, $\angle A_1OP = \theta$. Prove that

$$n\pi - \sum_{r=1}^{n} \angle OPA_r = \tan^{-1} \frac{a^n \sin n\theta}{a^n \cos n\theta - d^n}.$$

36. Show that

$$1 + \sum_{n=1}^{\infty} \frac{(-r^2)^n}{(2n)!} \cos 2n\theta = \cos (r\cos\theta) \cosh (r \sin\theta).$$

37. If $u + iv = \log\cos(x + iy)$, where $u$, $v$, $x$ and $y$ are real, prove that

$$u = \tfrac{1}{2}\log\{\tfrac{1}{2}(\cosh 2y + \cos 2x)\},$$

and that

$$\frac{\cos v}{\cos x\cosh y} = \frac{-\sin v}{\sin x\sinh y} = \frac{1}{\sqrt{\{\frac{1}{2}(\cosh 2y + \cos 2x)\}}}.$$

38. Prove that, if $n$ is a positive integer,

$$1 + x^n = \prod_{r=1}^{n}(1 - \alpha_r x),$$

where $\alpha_r = \exp\left(\frac{2r-1}{n}\pi i\right)$; then show that

$$\text{(i)} \sum_{r=1}^{n}\alpha_r{}^m = 0, \text{ if } m \text{ is an integer, not a multiple of } n,$$

$$\text{(ii)} \sum_{r=1}^{n}\alpha_r{}^{kn} = (-1)^k n, \text{ if } k \text{ is any integer.}$$

39. If $$1 - yx + x^2 = (1 - \alpha x)(1 - \alpha^{-1}x),$$
prove that

$$-\sum_{n=1}^{\infty}\frac{\alpha^n + \alpha^{-n}}{n}x^n = \log(1 - yx + x^2).$$

Deduce the expansion of $(\alpha^n + \alpha^{-n})$ in powers of $(\alpha + \alpha^{-1})$, and prove that

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \dots,$$

giving the general term; and, in the case of $n$ odd, the last term.

Ans. $$(-1)^r\frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!}(2\cos\theta)^{n-2r};$$

$$(-1)^{\frac{n-1}{2}}2n\cos\theta.$$

40. What is the principal value of $\log(1 + i\sqrt{3})$?

Ans. $\log 2 + i\frac{1}{3}\pi$.

41. If $-\pi < \theta < \pi$, show that

$$\log(1 + \cos\theta + i\sin\theta) = \log(2\cos\tfrac{1}{2}\theta) + i\tfrac{1}{2}\theta.$$

42. If $w = z^{1/2}\,\mathrm{Log}\,z$, and if, initially, $z = 1$, $z^{1/2} = 1$ and $\mathrm{Log}\,z = 2\pi i$, find the value of $w$ when $z$ returns to the point 1 after describing the circle $|z| = 1$ in the positive direction.

Ans. $-4\pi i$.

43. If $w = \log\left(\dfrac{1+z}{1-z}\right)$, and if $w = 0$ when $z = 0$, show that $w$ will be uniform if cross-cuts are taken along the $x$-axis from $-1$ to $-\infty$, and from $+1$ to $+\infty$. Find the value of $w$ for the values $+i$, $-i$, $1+i$ and $1-i$ of $z$.

Ans. $i\tfrac{1}{2}\pi, -i\tfrac{1}{2}\pi, \tfrac{1}{2}\log 5 + i(\tan^{-1}\tfrac{1}{2} + \tfrac{1}{2}\pi), \tfrac{1}{2}\log 5 - i(\tan^{-1}\tfrac{1}{2} + \tfrac{1}{2}\pi)$.

44. If $r$, $a$ and $\theta$ are real, $r$ positive, and $|a/r| < 1$, show that

$$\log(r^n - a^n\cos n\theta - ia^n\sin n\theta) = \sum_{s=0}^{n-1}\log\left\{r - a\cos\left(\theta + \frac{2s\pi}{n}\right) - ia\sin\left(\theta + \frac{2s\pi}{n}\right)\right\}.$$

45. If $\tan\alpha = \cos w\tan\theta$, prove that

$$\theta - \alpha = \sum_{n=1}^{\infty}\frac{1}{n}t^{2n}\sin 2n\alpha,$$

where $t = \tan\tfrac{1}{2}w$, $0 < w < \tfrac{1}{2}\pi$.

46. Find the sum of the series

$$n\sin\alpha + \tfrac{1}{2}n^2\sin 2\alpha + \tfrac{1}{3}n^3\sin 3\alpha + \dots,$$

where $-1 < n < 1$; and show that it is a solution of the equation

$$\sin x = n\sin(x + \alpha).$$

47. Establish the identities

$$\log\prod_{n=1}^{\infty}(1 - 2q^{2n-1}\cos 2x + q^{4n-2}) = \log\prod_{n=1}^{\infty}\{(1 - q^{2n-1}e^{2ix})(1 - q^{2n-1}e^{-2ix})\} = -\sum_{n=1}^{\infty}\frac{2}{n}\frac{q^n}{1-q^{2n}}\cos 2nx,$$

where $|q| < 1$; and show that these products and series are all absolutely convergent.

48. Express in the form $U + iV$,

$$\text{(i)} \ \tan\left\{\frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right)\right\},$$

$$\text{(ii)} \ e^{(a+ib)\,\text{Log}\,(x+iy)},$$

where $x, y, a, b$ are all real.

Ans. (i) $x$; (ii) $\exp\left[\frac{1}{2}a\log(x^2+y^2) - b\{\tan^{-1}(y/x) + m\pi\}\right] \times (\cos\phi + i\sin\phi)$, where $\phi = \frac{1}{2}b\log(x^2+y^2) + a\{\tan^{-1}(y/x) + m\pi\}$, and $m$ is zero or an even integer.

49. If $u + iv = \tan^{-1}(x + iy)$,
where $u, v, x, y$ are real, prove that

$$u = \tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}, \quad v = \tfrac{1}{4}\log\frac{(1+y)^2+x^2}{(1-y)^2+x^2}.$$

50. If
$$x + iy = \tan(u + iv),$$
where $x, y, u, v$ are real, show that

$$x = \frac{\sin 2u}{\cosh 2v + \cos 2u}, \qquad y = \frac{\sinh 2v}{\cosh 2v + \cos 2u},$$

$$\tan 2u = \frac{2x}{1-x^2-y^2}, \qquad \tanh 2v = \frac{2y}{1+x^2+y^2},$$

and then state the general value of $\tan^{-1}(x + iy)$.

51. If $\tanh y = \tan x$, and if, for small values of $x$ and $y$,
$$y = x + c_3x^3 + c_5x^5 + c_7x^7 + \dots,$$
show that
$$x = y - c_3y^3 + c_5y^5 - c_7y^7 + \dots.$$

52. If $-1 \leqq p \leqq 1$, show that

$$\frac{\cosh pz}{z\sinh z} = \frac{1}{z^2} - \frac{2\cos p\pi}{z^2+\pi^2} + \frac{2\cos 2p\pi}{z^2+4\pi^2} - \dots.$$

[If $n$ is a positive integer,

$$\left\{\left(1+\frac{pz}{2n+1}\right)^{2n+1} + \left(1-\frac{pz}{2n+1}\right)^{2n+1}\right\} \div \left\{\left(1+\frac{z}{2n+1}\right)^{2n+1} - \left(1-\frac{z}{2n+1}\right)^{2n+1}\right\}$$

$$= \frac{A}{z} + \sum_{r=1}^{n}\left\{\frac{A_r}{z - i(2n+1)\tan\dfrac{r\pi}{2n+1}} + \frac{B_r}{z + i(2n+1)\tan\dfrac{r\pi}{2n+1}}\right\}.$$

Here $A = 1$ while $A_r$ and $B_r$ are both equal to

$$\frac{\left(\cos\frac{r\pi}{2n+1}+ip\sin\frac{r\pi}{2n+1}\right)^{2n+1}+\left(\cos\frac{r\pi}{2n+1}-ip\sin\frac{r\pi}{2n+1}\right)^{2n+1}}{2\cos\frac{2nr\pi}{2n+1}\cos\frac{r\pi}{2n+1}}.$$

Thus the L.H.S., divided by $z$, is equal to

$$z^{-2}+\sum_{r=1}^{n}u_r\,,$$

where $u_r = (-1)^r \times$

$$\frac{\left(\cos\frac{r\pi}{2n+1}+ip\sin\frac{r\pi}{2n+1}\right)^{2n+1}+\left(\cos\frac{r\pi}{2n+1}-ip\sin\frac{r\pi}{2n+1}\right)^{2n+1}}{z^2\cos^2\frac{r\pi}{2n+1}+(2n+1)^2\sin^2\frac{r\pi}{2n+1}}$$

If $r$ is fixed and $n \to \infty$,

$$u_r \to (-1)^r\,\frac{\exp(ipr\pi)+\exp(-ipr\pi)}{z^2+r^2\pi^2} = (-1)^r\,\frac{2\cos pr\pi}{z^2+r^2\pi^2}.$$

Again, if $m$ is a positive integer such that $m > \frac{1}{2}|z|$, and if $r \geqq m$,

$$|u_r| < \frac{2\left(\cos^2\frac{r\pi}{2n+1}+p^2\sin^2\frac{r\pi}{2n+1}\right)^{n+\frac{1}{2}}}{r^2\pi^2\left(\dfrac{\sin\frac{r\pi}{2n+1}}{\frac{r\pi}{2n+1}}\right)^2-\left|z\cos\frac{r\pi}{2n+1}\right|^2} < \frac{2}{4r^2-|z|^2}.$$

The result then follows by means of Tannery's Theorem.]

53. Show that, if $-\pi \leqq a \leqq \pi$,

$$\pi\frac{\cos az}{\sin \pi z} = \frac{1}{z}+2z\sum_{n=1}^{\infty}(-1)^n\frac{\cos na}{z^2-n^2}.$$

54. If $-1 < \lambda < 1$, show that

$$\frac{\sin \lambda u}{\sin u} = \sum_{n=1}^{\infty}(-1)^n\frac{2n\pi\sin n\pi\lambda}{u^2-n^2\pi^2}.$$

55. If

$$S_m = \frac{c}{a^2 + c^2} + \sum_{n=1}^{m} \left\{ \frac{n + c}{(n + c)^2 + a^2} - \frac{n - c}{(n - c)^2 + a^2} \right\},$$

apply the expansion

$$\pi \coth \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2}$$

to prove that

$$\mathop{\mathcal{L}}_{m \to \infty} S_m = \frac{\pi \sin 2\pi c}{\cosh 2\pi a - \cos 2\pi c}.$$

56. Show that the principal value of $i^i$ is $e^{-\frac{1}{2}\pi}$.

57. Show that, if $|\tan x| < 1$,

(i) $\cos nx = (\cos x)^n \, F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}; -\tan^2 x\right)$,

(ii) $\sin nx = n \sin x (\cos x)^{n-1} \, F\left(\frac{1-n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; -\tan^2 x\right)$,

(iii) $\cos nx (\cos x)^n = F\left(\frac{n}{2}, \frac{1+n}{2}; \frac{1}{2}; -\tan^2 x\right)$,

(iv) $\sin nx (\cos x)^n = n \tan x \, F\left(\frac{1+n}{2}, 1 + \frac{n}{2}; \frac{3}{2}; -\tan^2 x\right)$.

[From Demoivre's Theorem

$$\cos nx = \tfrac{1}{2}(\cos x + i \sin x)^n + \tfrac{1}{2}(\cos x - i \sin x)^n$$
$$= \tfrac{1}{2}(\cos x)^n \{(1 + i \tan x)^n + (1 - i \tan x)^n\}.$$

Now expand by the binomial theorem to get (i). For (ii) use the formula

$$\sin nx = \frac{1}{2i}(\cos x + i \sin x)^n - \frac{1}{2i}(\cos x - i \sin x)^n.$$

For (iii) and (iv) put $-n$ for $n$ in (i) and (ii).]

58. If $(1 + z)^n = p_0 + p_1 z + p_2 z^2 + \ldots$, show that

(i) $p_0 - p_2 + p_4 - \ldots = 2^{\frac{1}{2}n} \cos \frac{1}{4}n\pi$,

(ii) $p_1 - p_3 + p_5 - \ldots = 2^{\frac{1}{2}n} \sin \frac{1}{4}n\pi$.

59. Show that, if $|z| < 1$,

$$1 + 2z + 3z^2 + 4z^3 + \ldots = (1 - z)^{-2},$$

and then sum the series

(i) $1 + 2r\cos\theta + 3r^2\cos 2\theta + 4r^3\cos 3\theta + \ldots$,
(ii) $2r\sin\theta + 3r^2\sin 2\theta + 4r^3\sin 3\theta + \ldots$,
(iii) $\sin\theta + 2r\sin 2\theta + 3r^2\sin 3\theta + 4r^3\sin 4\theta + \ldots$,

where $-1 < r < 1$.

Ans. (i) $(1 - 2r\cos\theta + r^2\cos 2\theta)/(1 - 2r\cos\theta + r^2)^2$,
(ii) $(2r\sin\theta - r^2\sin 2\theta)/(1 - 2r\cos\theta + r^2)^2$,
(iii) $(1 - r^2)\sin\theta/(1 - 2r\cos\theta + r^2)^2$.

60. If $z_n = \cos\dfrac{\pi}{2^n} + i\sin\dfrac{\pi}{2^n}$, show that

$$\prod_{n=1}^{\infty} z_n = -1.$$

61. If $z_n = \cos\dfrac{\pi}{3^n} + i\sin\dfrac{\pi}{3^n}$, show that

$$\prod_{n=1}^{\infty} z_n = i.$$

62. Show that, if $\alpha$ is neither zero nor an integral multiple of $2\pi$,

$$\frac{\cosh x - \cos\alpha}{1 - \cos\alpha} = \prod_{n=-\infty}^{\infty}\left\{1 + \frac{x^2}{(2n\pi + \alpha)^2}\right\}.$$

63. Prove that

$$\frac{\left(1 + \frac{4}{\pi^2}\right)\left(1 + \frac{4}{9\pi^2}\right)\left(1 + \frac{4}{25\pi^2}\right)\cdots}{\left(1 + \frac{1}{\pi^2}\right)\left(1 + \frac{1}{4\pi^2}\right)\left(1 + \frac{1}{9\pi^2}\right)\cdots} = \frac{e^2 + 1}{e^2 - 1}.$$

64. Show that

$$1 + \frac{2x^2}{1 + x^2} + \frac{2x^2}{2^2 + x^2} + \frac{2x^2}{3^2 + x^2} + \cdots = \frac{(1 + 4x^2)\left(1 + \frac{4x^2}{3^2}\right)\left(1 + \frac{4x^2}{5^2}\right)\cdots}{(1 + x^2)\left(1 + \frac{x^2}{2^2}\right)\left(1 + \frac{x^2}{3^2}\right)\cdots}.$$

65. Show that

$$\text{(i)}\ \cos z = \prod_{n=1}^{\infty} \left\{1 - \frac{4z^2}{(2n-1)^2\pi^2}\right\},$$

$$\text{(ii)}\ \sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2\pi^2}\right),$$

$$\text{(iii)}\ \cosh z = \prod_{n=1}^{\infty} \left\{1 + \frac{4z^2}{(2n-1)^2\pi^2}\right\}.$$

66. Prove that

$$\sin \pi z = \pi z \prod_{n=-\infty}^{\infty}{}' \left\{\left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}}\right\}.$$

67. Show that

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^4}{n^4}\right) = \frac{\cosh(\pi x\sqrt{2}) - \cos(\pi x\sqrt{2})}{2\pi^2 x^2}.$$

68. Show that

$$\cos z = \prod_{r=0}^{n-1} \left\{1 - \sin^2 \frac{z}{2n} \Big/ \sin^2 \frac{(2r+1)\pi}{4n}\right\},$$

and deduce that

$$\cos z = \prod_{r=0}^{\infty} \left\{1 - \frac{4z^2}{(2r+1)^2\pi^2}\right\}.$$

69. In formula (30), page 459, put $\theta = \pi$ and deduce that, if $\alpha$ is not integral,

$$\frac{\pi}{\sin \alpha\pi} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} (-1)^n \frac{2\alpha}{\alpha^2 - n^2}.$$

# APPENDIX

## The Length of a Circular Arc

In Chapter I it was assumed that an arc of a circle has a definite length, an assumption which requires justification. A definition of the length of a circular arc, based on the conception of the length of a straight line, will now be given.

In what follows it is assumed that the arc considered is less than a semi-circle. When once the length of such an arc has been defined, the lengths of larger arcs can be obtained by addition.

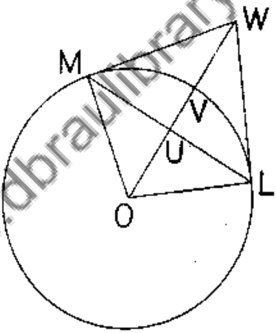


Fig. 1.

Let LM (Fig. 1) be an arc of a circle whose centre is O, and let the tangents at L and M meet in W. Let OW cut the chord LM and the arc LM in U and V respectively. Then

$$\frac{LW}{LU} = \frac{OL}{OU},$$

and, consequently,

$$\frac{LW - LU}{LW} = \frac{OL - OU}{OL} = \frac{OV - OU}{OL} = \frac{UV}{OV}.$$

Hence $$LW - LU = \frac{UV}{OV} LW;$$

and, similarly, $$WM - UM = \frac{UV}{OV} WM.$$

Thus, on adding, we have

$$(LW + WM) - LM = \frac{UV}{OV}(LW + WM). \quad . \quad (1)$$

Again, let N (Fig. 2) be any point on the arc LM, and let the tangent at N meet LW and WM in R and S respectively. Then

$$LN + NM > LM \quad . \quad . \quad . \quad (2)$$

and $LR + RN + NS + SM = LR + RS + SM$
$$< LR + (RW + WS) + SM,$$

so that $$LR + RN + NS + SM < LW + WM. \quad . \quad (3)$$

Next, let AB (Fig. 3) be an arc of a circle, and let the tangents at A and B meet in T. Take $n - 1$ points $K_1$, $K_2$, . . . , $K_{n-1}$ in order on the arc AB, and let the tangents at the pairs

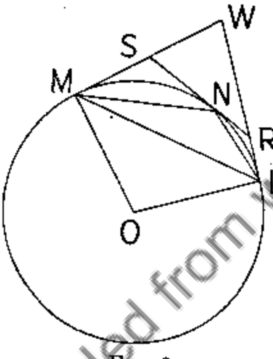


Fig. 2.

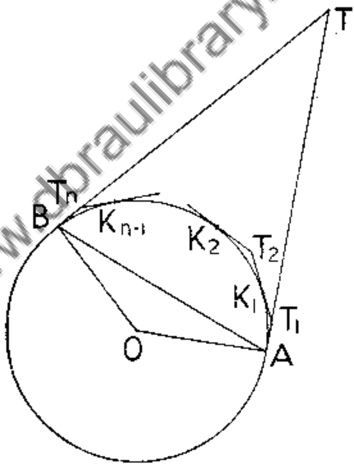


Fig. 3.

of points A, $K_1$; $K_1$, $K_2$; . . . ; $K_{n-1}$, B meet in $T_1$, $T_2$ . . . , $T_n$ respectively. Also let

$$S_n = AK_1 + K_1K_2 + \ldots + K_{n-1}B,$$

and $$\Sigma_n = AT_1 + T_1T_2 + \ldots + T_nB,$$

$S_n$ and $\Sigma_n$ being the sums of chords and tangents respectively.

Then, if $n$ be increased by the insertion of additional points $K_r$ on the arc AB, it is clear from (2) and (3) that $S_n$ increases and $\Sigma_n$ decreases as $n$ increases. Also from Fig. 3 it is clear that $S_n < \Sigma_n$, so that

$$AB < S_n < \Sigma_n < AT + TB.$$

Thus (Ch. XVIII, § 1, Theorem I), since $S_n$ increases with $n$ and

$$S_n < AT + TB,$$

$S_n$ tends to a definite limit as $n \to \infty$. Similarly (Ch. XVIII, § 1, Theorem II), since $\Sigma_n$ decreases as $n$ increases and

$$\Sigma_n > AB,$$

$\Sigma_n$ tends to a definite limit as $n \to \infty$.

If it be now further assumed that the greatest of the chords $K_{r-1}K_r$ tends to zero as $n \to \infty$, it can be shown that these two limits are equal.

For, from (1),

$$(K_{r-1}T_r + T_rK_r) - K_{r-1}K_r = \lambda_r(K_{r-1}T_r + T_rK_r),$$

where $\lambda_r \to 0$ when $K_{r-1}K_r \to 0$. Let $\lambda$ be the greatest of $\lambda_1, \lambda_2, \ldots, \lambda_n$; then

$$(K_{r-1}T_r + T_rK_r) - K_{r-1}K_r \leqq \lambda(K_{r-1}T_r + T_rK_r).$$

Thus, on adding the corresponding inequalities for

$$r = 1, 2, \ldots, n,$$

it is found that

$$\Sigma_n - S_n \leqq \lambda\Sigma_n < \lambda(AT + TB).$$

But, when $n \to \infty$, $\lambda \to 0$; therefore

$$\Sigma_n - S_n \to 0.$$

Hence the limits of $\Sigma_n$ and $S_n$ are identical. Denote this common limit by $l$.

It will next be proved that the value of $l$ is the same no matter how the points of division are selected, provided only that the length of the greatest chord tends to zero as $n$ tends to infinity.

Let a set of $m$ points (of which some or all are different from the $n$ points) be taken on the arc AB, and let $S'_m$ and $\Sigma'_m$ be the sums corresponding to $S_n$ and $\Sigma_n$. It is assumed that the length of the greatest chord tends to zero as $m \to \infty$, so that $S'_m$ and $\Sigma'_m$ tend to a common limit $l'$. Let $n$ and $m$ be taken so large that

$$l - \tfrac{1}{2}\epsilon < S_n < \Sigma_n < l + \tfrac{1}{2}\epsilon,$$

and

$$l' - \tfrac{1}{2}\epsilon < S'_m < \Sigma'_m < l' + \tfrac{1}{2}\epsilon,$$

where $\epsilon$ is an arbitrarily assigned positive quantity.

Now, superimpose the one set of divisions of the arc AB on the other, and let $s_{n+m}$, $\sigma_{n+m}$ denote the sum of the chords and the sum of the tangents so obtained. Then

$$l - \tfrac{1}{2}\epsilon < S_n < s_{n+m} < \sigma_{n+m} < \Sigma_n < l + \tfrac{1}{2}\epsilon$$

and $$l' - \tfrac{1}{2}\epsilon < S'_m < s_{n+m} < \sigma_{n+m} < \Sigma'_m < l' + \tfrac{1}{2}\epsilon;$$

so that $$l - \tfrac{1}{2}\epsilon < l' + \tfrac{1}{2}\epsilon,$$

and $$l' - \tfrac{1}{2}\epsilon < l + \tfrac{1}{2}\epsilon.$$

Thus $l - l' < \epsilon$ and $l' - l < \epsilon$, and consequently $l - l'$ is numerically less than $\epsilon$. But $\epsilon$ can be chosen as small as we please: therefore $l$ and $l'$ are equal.

It follows that, no matter what points $K_1, K_2, \ldots, K_{n-1}$ are taken on the arc AB, the sums $S_n$ and $\Sigma_n$ tend to one definite limit as $n \to \infty$, provided only that the greatest of the chords tends at the same time to zero. This limit is taken to be the length of the arc AB.

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